1. Find a 1-to-1 conformal map from the sector $S_{\frac{\pi}{4}} = \{z : 0 < \arg z < \frac{\pi}{4}\}$ onto the interior of the unit disk $D_1 = \{z : |z| < 1\}$.

2. Let $<f_n>$ be a sequence of functions analytic in the unit disk, $D_1$, and let $f$ be a continuous function also defined in $D_1$. Show that if

$$\lim_{n \to \infty} \iint_{D_1} |f_n(x, y) - f(x, y)| dxdy = 0,$$

then the function $f$ is also analytic in $D_1$.

3. Suppose that $A$ is an $n \times n$ matrix that commutes with all $n \times n$ diagonal matrices. What can we say about $A$; you must prove your answer.

4. (a) What is a Hermitian inner product?
   (b) What is unitary matrix?
   (c) What is a Hermitian matrix?
   (d) Suppose that $A$ is an Hermitian matrix. Prove that $(A - i \text{Id})$ is invertible and the matrix $Z = (A + i \text{Id})(A - i \text{Id})^{-1}$ is a unitary matrix.

5. Show that there is no non-constant polynomial $P(u, v)$ in two variables such that

$$P(x, \cos x) = 0$$

holds for all $x \in \mathbb{R}$.

6. Consider the equation $ye^y = x$. Show that this uniquely defines a function $y(x)$ on the interval $[0, \infty)$.
   (a) Sketch the graph of $y(x)$.
   (b) Find a formula for $\frac{dy}{dx}$ as a function of $y$.
   (c) Give a method to calculate the value of $y$ such that $ye^y = 1$, to arbitrary accuracy.
7. Fix a large number $N$ and for $1 \leq j \leq N$ let $\{U_j\}$ be IID random variables uniform on the interval $[0, 1]$. Let $X$ be the number of these that are local maxima, that is, the number of $j$ such that $U_{j-1} < U_j > U_{j+1}$. The indices are considered modulo $N$, e.g., a maximum occurs at 1 if $U_N < U_1 > U_2$.

(a) Compute the expected value $\mu := E(X)$.
(b) Compute the second moment $M := E(X^2)$.
(c) Compute the variance $V := \text{Var}(X)$.
(d) For what real number $q$ can you prove a nontrivial upper bound on

$$\text{Prob}(|X - \mu| > cN^q)$$

that does not depend on $N$? It will of course depend on $c$, and nontrivial means it has to be less than 1 for at least a range of values of $c$. You are not asked to get the best bound, just any bound uniform in $N$. 