1. For which values of $x$ is the following series convergent

\[ \sum_{n=1}^{\infty} \frac{\cos(nx)}{n} \]

How about

\[ \sum_{n=1}^{\infty} \frac{\cos(nx)}{n^3} \]

2. Let $f(z)$ be analytic in $|z| < 1$ and suppose that $f(0) = 0$. Prove that

\[ G(z) = \sum_{n=1}^{\infty} f(z^n) \]

is also analytic in $|z| < 1$. You can use the fact that a uniformly convergent sequence of analytic functions has an analytic limit. If the limit $G(z)$ is constant, then what can you say about $f$? You must justify your answer.

3. Let $V$ be an $(n+1)$-dimensional real vector space of real valued functions defined on the interval $[0, 1]$. Show that if $0 \leq x_1 < \cdots < x_n \leq 1$ are arbitrary, then there is a function $f \in V$ such that

\[ f(x_i) = 0 \text{ for } i = 1, \ldots, n. \]

4. Suppose that $(a_{i,j})$ is a positive definite, symmetric $n \times n$ matrix, that is

\[ A(u) = \sum_{i,j=1}^{n} a_{i,j} u_i u_j \geq 0, \]

and $A(u) = 0$ only if $u = 0$. Suppose that $(f_1, \ldots, f_n) \neq (0, \ldots, 0)$ is real vector. Show that the function

\[ G(u) = A(u) + \sum_{j=1}^{n} f_j u_j \]

attains its minimum value. Show that this minimum value is negative and find equations to determine the vector $u$ where this value is attained. What is the minimum value?
5. Let $\mathcal{P}_n$ denote the vector space of polynomials of degree at most $n$. Prove that, for each $n$, the differential operator

\begin{equation}
L = x(1 - x)\partial_x^2 + \frac{1}{2}x\partial_x
\end{equation}

maps $\mathcal{P}_n$ to $\mathcal{P}_n$. Show that, for each $n$, $L : \mathcal{P}_n \rightarrow \mathcal{P}_n$ is diagonalizable and find its spectrum. Hint: Consider the operator $-x^2\partial_x^2 + \frac{1}{2}x\partial_x$.

6. Suppose that two buses arrive independently at a bus stop so that, for $0 \leq t$, the arrival times satisfy

\begin{equation}
\text{Prob}(t \leq X < t + dt) = \lambda e^{-\lambda t} dt \quad \text{and} \quad \text{Prob}(t \leq Y < t + dt) = \mu e^{-\mu t} dt,
\end{equation}

where $\lambda$ and $\mu$ are positive constants. What is the

\begin{equation}
\text{Prob}(\min\{X, Y\}) < T?
\end{equation}

7. A fair coin is flipped repeatedly. What is the expected value for the number of flips needed to first see two HEADS in a row? Hint: Write formulas for the expected values conditional on the outcome of the last flip.