AMCS Written Preliminary Exam  
Part I, May 1, 2017

1. Let \( f(x) \) be a non-negative, monotone decreasing function for which the integral 
\[
\int_{0}^{\infty} f(x) \, dx < \infty.
\]
Prove that 
\[
\lim_{x \to \infty} x f(x) = 0.
\]

2. Find three different Laurent expansion for function 
\[
f(z) = \frac{1}{1 + 3z + z^2}.
\]
State where each expansion is valid.

3. Let \( A \) be the \( 9 \times 9 \) matrix 
\[
A = \begin{pmatrix}
2s & t & t & t & \cdots & t \\
t & 2s & t & t & \cdots & t \\
t & t & 2s & t & \cdots & t \\
\vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\
t & t & t & t & \cdots & 2s
\end{pmatrix}
\]
(All off-diagonal entries are \( t \) and all diagonal entries are \( 2s \).) For which complex values of \( t \) and \( s \) is this matrix invertible?

4. Find the maximum of \( x_1^2 \cdot x_2^2 \cdots x_n^2 \) subject to the constraint 
\[
x_1^2 + \cdots + x_n^2 = 1.
\]
From the solution to this problem prove that, for arbitrary positive real numbers \( r_1, \ldots, r_n \), we have the inequality 
\[
(r_1 \cdots r_n)^{\frac{1}{n}} \leq \frac{r_1 + \cdots + r_n}{n}.
\]
Prove this directly, without recourse to induction.
5. Suppose that $A$ is an invertible $n \times n$ matrix with characteristic polynomial

$$\det(A - \lambda \text{Id}) = \sum_{j=0}^{n} a_j \lambda^j.$$ 

What is the characteristic polynomial of $A^{-1}$? If $B$ is a $2 \times 2$ matrix with characteristic polynomial

$$\det(B - \lambda \text{Id}) = \lambda^2 - 3\lambda - 3,$$

then what is the characteristic polynomial of $2B - 3\text{Id}$?

6. Let $X$ and $Y$ be independent, standard normal, random variables, that is, their joint density is

$$p(x, y) = \frac{1}{2\pi} e^{-\frac{x^2+y^2}{2}}.$$ 

Give an expression for $\text{Prob}(X + Y < -1)$ in terms of the normal CDF

$$\Phi(x) := \int_{-\infty}^{x} \frac{e^{-t^2/2}}{\sqrt{2\pi}} dt.$$ 

7. A simple random walk is a particle moving, at each time step, either left or right by 1 unit, each with probability 1/2. What is the average amount of time it takes a simple random walk, started at 1, to hit the set $\{0, 5\}$?