1. Prove that
   \[
   \log 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots
   \]

2. Suppose that \( f(z) \) is an analytic function in \( D_R = \{ z : |z| < R \} \) such that \( |f(z)| < M \) for \( z \in D_R \), and \( f(0) = 1 \). Find a number \( 0 < \rho < R \), such that \( f(z) \neq 0 \) for any \( z \) with \( |z| \leq \rho \).

3. There is an orthogonal transformation \( O \) of \( \mathbb{R}^3 \) that transforms the quadratic form
   \[
   q(x, y, z) = 2xy + 2xz + 2yz
   \]
to the quadratic form
   \[
   Q(X, Y, Z) = \lambda_1 X^2 + \lambda_2 Y^2 + \lambda_3 Z^2.
   \]
   Here \( (X, Y, Z) \) are the transformed variables. What are the values of \( \lambda_1, \lambda_2, \lambda_3 \).

4. Let \( \{a_1, \ldots, a_n\} \) and \( c \) be positive numbers with \( c > n \). Use Lagrange multipliers to show that the minimum value of
   \[
   f(x) = \sum_{j=1}^{n} \frac{a_j}{x_j},
   \]
on the set
   \[
   S = \left\{ 0 \leq x_j \leq 1 : \sum_{j=1}^{n} \frac{1}{1 - x_j} = c \right\},
   \]
is
   \[
   \sum_{j=1}^{n} a_j + \frac{1}{c - n} \left( \sum_{j=1}^{n} a_j \frac{1}{2} \right)^2.
   \]

5. Suppose that \( g(z) \) is an entire function that never vanishes. What are all the possible values of the integrals
   \[
   \int_C \frac{1}{zg(z)} \, dz,
   \]
   where \( C \) is any smooth curve that does not pass through 0 and goes from \( z = 1 \) to \( z = z_0 \).

6. In a set of 4000 independent fair coin flips, what is the probability of getting 3000 or more HEADS? Please answer to within a factor of 10. The following common logarithms are accurate to roughly one part in 4000: \( \log 2 = 0.301, \log 3 = 0.477 \).
7. Let $A$ be a non-singular square matrix ($\det A \neq 0$). Show that there is a polynomial, $p(\lambda) = c_k \lambda^k + \cdots + c_1 \lambda + c_0$ such that

$$A^{-1} = c_0 \text{Id} + c_1 A + \cdots + c_k A^k.$$