AMCS Written Preliminary Exam
Part II, May 8, 2013

1. What is the radius of convergence of the power series around $z = 0$ for the function

$$f(z) = \frac{z - i}{(z^2 - \frac{1}{4})(z + i)}? \tag{1}$$

   How about the power series centered at $z = 2i$?

2. Find a conformal map from the half strip

$$D = \{z : \operatorname{Re} z < 0 \text{ and } 0 < \operatorname{Im} z < \pi\} \tag{2}$$

   to the upper half plane $H_+ = \{z : \operatorname{Im} z > 0\}$.

3. Compute the following integrals:

   (a)

$$\int_0^\infty \frac{\log x \, dx}{1 + x^2},$$

   (b)

$$\int_{-\infty}^\infty \frac{\cos(ax)dx}{x^2 + a^2}, \text{ for } a \in (0, \infty).$$

4. If $A = (a_{ij})$ is an $n \times n$ matrix, then the trace of A, $\operatorname{tr} A$ is defined to be

$$\operatorname{tr} A = \sum_{i=1}^n a_{ii}. \tag{3}$$

   Show that if A and B are two $n \times n$ matrices then

$$\operatorname{tr} AB = \operatorname{tr} BA. \tag{4}$$

   Find a formula for the $\operatorname{tr} A$ in terms of the eigenvalues of A and prove that it is correct.

5. Suppose that X is a 3×3 real, skew symmetric matrix, that is $X^t = -X$. Prove that there is a non-zero vector v such that $Xv = 0$.

6. Let $\mathcal{P}_3$ denote polynomials of degree at most 3. The differential operator

$$Lp = x\partial_x^2 p - 2\partial_x p \tag{5}$$

   defines a linear map from $\mathcal{P}_3$ to itself. What is the matrix of this transformation in terms of the basis $\{1, x, x^2, x^3\}$? What is the rank of L? Give a basis for the range of L and the nullspace of L.

7. Suppose that f is a 2π-periodic function that satisfies the estimate

$$|f(x) - f(y)| \leq M|x-y|^{\alpha}$$

for an $0 < \alpha < 1$, and let

$$\hat{f}(n) = \frac{1}{2\pi}\int_0^{2\pi} f(x)e^{-inx}\,dx. \tag{6}$$

Show that

$$S_N(x) = \sum_{n=-N}^{N} \hat{f}(n)e^{inx} \tag{7}$$

converges uniformly to $f(x)$ for all real x. Hint:

$$S_N(x) = \int_0^{2\pi} f(y)\frac{\sin(N+\frac{1}{2})(x-y)}{2\pi\,\sin(\frac{1}{2}(x-y))}dy. \tag{8}$$

8. Suppose that X_1 and X_2 are independent random variables, uniformly distributed in $[-1, 1]$. What is the probability that $\max\{X_1, X_2\} \geq 0$.