AMCS Written Preliminary Exam
Part I, May 8, 2013

1. Let A and B be subsets of $\mathbb{R}$, such that for any $y \in B$ there exists a $x \in A$ with $x \leq y$. Show that

$$\inf A \leq \inf B. \tag{1}$$

2. Define the function

$$f(x) = \begin{cases} \frac{1}{q} \text{ if } x = \frac{p}{q} \text{ with } (p,q) = 1 \\ 1 \text{ if } x \notin \mathbb{Q}. \end{cases} \tag{2}$$

   Is f Riemann integrable on $[0, 1]$? Why or why not?

3. Show that the series

$$S = \sum_{j=1}^{\infty} (-1)^n \log\left(1 + \frac{1}{n}\right) \tag{3}$$

   is convergent. If we let

$$S_N = \sum_{j=1}^{N} (-1)^n \log\left(1 + \frac{1}{n}\right), \tag{4}$$

   then how large must N_0 be so that $|S - S_N| \leq 10^{-4}$ if $N \geq N_0$.

4. Suppose that A is an $n \times n$ matrix and there exists a matrix B so that

$$AB = \text{Id}, \tag{5}$$

   Prove that A is invertible and $BA = \text{Id}$, as well.

5. Let A be the 3×3 matrix:

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & x \\ x & 0 & y \end{bmatrix}.$$

   For which pairs (x, y) is the equation $Au = v$ solvable for any v? When the equation is not solvable for some v, what is the condition for solvability?

6. A real valued $\mathscr{C}^2$-function u, defined in the unit disk, D_1 is harmonic if it satisfies the partial differential equation $\partial_{xx} u + \partial_{yy} u = 0$. Prove that a $\mathscr{C}^2$-function u defined in D_1 is harmonic if and only if for each $(x, y) \in D_1$

$$u(x, y) = \frac{1}{2\pi} \int_0^{2\pi} u(x + r\cos\theta, y + r\sin\theta) d\theta, \tag{6}$$

for sufficiently small positive r. Hint: Recall Green's formula: For a bounded region D with $\mathscr{C}^1$ boundary, and a $\mathscr{C}^2$ function v defined in $\overline{D}$, we have.

$$\int\limits_D \Delta v \, dA = \int\limits_{bD} \partial_\nu v \, ds \tag{7}$$

Here ν is the outer unit normal along bD.

7. Suppose that a fair coin is flipped $2n$ times. What is the probability that you get an equal number of heads and tails?
8. Let $\{Y_n\}$ be independent random variables uniformly distributed in $[0, 1]$. For $n \geq 1$, let $X_n = \exp(Y_1) \cdots \exp(Y_n)$. Compute $E[X_n]$.