AMCS Written Preliminary Exam, II
August 30, 2011

All work should go in the exam booklet, with your final answer clearly marked.

1. Evaluate the integral
$$\int_0^\infty \frac{\log x\,dx}{x^2+a^2}.$$
You must show that this improper Riemann integral exists.

2. For $a > 1$, show that:
$$\int_0^\pi \frac{dx}{a+\cos x} = \frac{\pi}{\sqrt{a^2-1}}.$$

3. Suppose that f, g are analytic functions in the disk $D_1(0)$, and that $\bar{f}g$ is also analytic. Show that either f is constant or g is zero.

4. Let $< A_n >$ be a sequence of square matrices converging to A. Give a proof, or counterexample for the following statements:
   (a) If each A_n is singluar, then A is singular.
   (b) If each A_n is non-singluar, then A is non-singular.

5. Let A be 2×2 real matrix and set
$$r(A) = \max_{\boldsymbol{x} \neq 0} \frac{\|A\boldsymbol{x}\|}{\|\boldsymbol{x}\|}, \tag{1}$$
where $\|\cdot\|$ is the Euclidean norm. Does the matrix A always have an eigenvalue λ with $|\lambda| = r(A)$? Give a proof, or counterexample.

6. A stick of length 1 is broken into three pieces randomly. What is the probability that these pieces are the edges a triangle? The edge lengths are given by triples:
$$S = \{(x_1, x_2, x_3) : x_1, x_2, x_3 \geq 0 \text{ with } x_1 + x_2 + x_3 = 1\},$$
The probability distribution on edge lengths is defined by the surface measure on S, normalized to have total area 1.

7. Two points are chosen independently and uniformly from the unit interval. What is the expected square of the distance between them?

8. Fix an even number, n. The random variables $X_1, X_2, ..., X_n$ take values in the set $\{-1, 1\}$. There is a constant c so that, for $i \neq j$, $E[X_i X_j] = c$. Find the sharpest lower bound you can for c. Hint: If Y is a non-negative random variable, then $E[Y] \geq 0$.

9. What is the expected number of times that a standard (fair), six-sided die must be rolled so that it lands on each face at least once?