AMCS Written Preliminary Exam, I
August 30, 2011

All work should go in the exam booklet, with your final answer clearly marked.

1. Choose a number $x_0 \in [1, 4]$ and define $x_{n+1} = \frac{2+2x_n^3}{3x_n^2}$ for $n \in 0, 1, 2, \ldots$ Find the limit
$$L = \lim_{n\to\infty} x_n,$$
and prove that $< x_n >$ converges to L.

2. Suppose that $\{F_j\}$ are closed bounded subsets of $\mathbb{R}^n$, and G is an open subset. Show that if
$$\bigcap_{j=1}^{\infty} F_j \subset G,$$
then there is a finite subset $\{j_1, \ldots, j_k\}$ so that
$$F_{j_1} \cap \cdots \cap F_{j_k} \subset G.$$

3. A real valued function defined on (a, b) is said to be convex if for $x, y \in (a, b)$ and $\lambda \in (0, 1)$ we have the estimate:
$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y).$$
   (a) Prove that a convex function is continuous.
   (b) Prove that a bounded convex, differentiable function defined on $\mathbb{R}$ is constant.

4. Suppose that $0 < \alpha < 1$. Show that there is a constant C_α so that for $x, y \in (0, \infty)$ we have the estimate
$$\frac{1}{C_\alpha}(x^\alpha + y^\alpha) \leq (x+y)^\alpha \leq C_\alpha(x^\alpha + y^\alpha).$$

5. If v and w are vectors in $\mathbb{R}^n$, then the linear transformation $v \otimes w^t$ is defined by

$$v \otimes w^t \cdot x = \langle x, w \rangle v,$$

where $\langle \cdot, \cdot \rangle$ in an inner product on $\mathbb{R}^n$. Show that if A is an $n \times n$ matrix of rank m, then there are m pairs $\{(v_i, w_i) : i = 1, \ldots, m\}$ so that

$$A = \sum_{i=1}^{m} v_i \otimes w_i^t.$$

6. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation such that $T(x_1, x_2) = (x_2, x_1)$.

   (a) Show that T is self adjoint with respect to the standard inner product, $\langle \boldsymbol{x}, \boldsymbol{y} \rangle = x_1 y_1 + x_2 y_2$, on $\mathbb{R}^2$.

   (b) Define a new inner product $\langle \boldsymbol{x}, \boldsymbol{y} \rangle'$ so that

   $$\langle \boldsymbol{x}, \boldsymbol{x} \rangle' = x_1^2 + x_1 x_2 + \frac{1}{3} x_2^2.$$

   Write the matrix for this inner product, i.e. the symmetric 2×2 matrix so that

   $$\langle \boldsymbol{x}, \boldsymbol{y} \rangle' = \langle A\boldsymbol{x}, \boldsymbol{y} \rangle.$$

   (c) Find T^* with respect to the new inner product.

7. Suppose that f is a real, continuously differentiable function on $[0, 1]$.

   Prove that for $0 \leq a < b \leq 1$, we have the estimate

   $$|f(b) - f(a)| \leq \sqrt{|b - a|} \left[\int_0^1 |f'(x)|^2 dx \right]^{\frac{1}{2}}.$$

   Let $< f_n >$ be a sequence of continuously differentiable functions on $[0, 1]$ for which $< f_n(0) >$ is a bounded sequence, and there exists an M such that

   $$\int_0^1 |f_n'(x)|^2 dx < M.$$

   Show that $< f_n >$ has a uniformly convergent subsequence. Is the limit necessarily differentiable?