AMCS Written Preliminary Exam, II
August 31, 2010

All work should go in the exam booklet, with your final answer clearly marked.

1. Evaluate the integral
$$\int_0^\infty \frac{x \sin x dx}{x^2 + a^2}.$$
You must show that this improper Riemann integral exists.

2. Evaluate the integral
$$\int_0^\infty \frac{dx}{1 + x^{1000}}.$$

3. Suppose that the function $f(x, y) = u(x, y) + iv(x, y)$ is analytic in the unit square, $(0, 1) \times (0, 1)$, and that $f'(x, y) \neq 0$, for any (x, y) in the square. For $a, b \in \mathbb{R}$ let
$$l_a = \{(x, y) : \ u(x, y) = a\} \text{ and } m_b = \{(x, y) : \ v(x, y) = b\}.$$
Suppose that for some choice of (a, b), the set $I_{ab} = l_a \cap m_b \neq \emptyset$. Show that
   (a) I_{ab} is a discrete subset of the unit square.
   (b) At each point $(x, y) \in I_{ab}$ the curve l_a meets the curve m_b at right angles.
   (c) The curves l_a and m_b are smooth and do not have self-intersections.

4. Let $A = (a_{ij})$ be an $n \times n$ complex matrix such that for each $i = 1, \ldots, n$ we have
$$\sum_{j \neq i} |a_{ij}| < |a_{ii}|.$$
Can the matrix A be singular? Why or why not?

5. Find numbers a, b, c so that the following system of equations has *no* solution:
$$\begin{aligned} x - y + 2z &= a \\ 2x + 2z &= b \\ x - 3y + 4z &= c \end{aligned}$$
You must justify your answer.

6. You are waiting at a bus stop. Bus A has an arrival rate of μ; bus B has an arrival rate of ν, and they arrive independently. What is the probability that Bus A arrives before Bus B?

7. A fair coin is flipped repeatedly. What is the expected number of flips needed to see two HEADS in a row?

8. If X and Y are independent random variables which are uniformly distributed on the interval [0, 1], what is the probability that $|X - Y| > 1/2$?