AMCS Written Preliminary Exam, I
August 31, 2010

All work should go in the exam booklet, with your final answer clearly marked.

1. Choose a number $x_0 \in [0, \pi)$ and define $x_{n+1} = \sin(x_n)$ for $n \in 0, 1, 2, \ldots$ Evaluate
$$L = \lim_{n\to\infty} x_n. \tag{1}$$
You must prove your answer.

2. Suppose that f is a real, continuously differentiable function on $[0, 1]$ with $f(0) = f(1) = 0$, and
$$\int_0^1 [f(x)]^2 dx = 1.$$
Prove that
$$\int_0^1 x f(x)\, f'(x) dx = -\frac{1}{2}$$
and
$$\frac{1}{4} < \int_0^1 x^2 [f(x)]^2 dx \cdot \int_0^1 [f'(x)]^2 dx$$

3. Give a rigorous proof that the following limit exists:
$$\lim_{R\to\infty} \int_0^R e^{ix^2} dx.$$

4. A rod of length 1 has endpoints constrained to lie on the positive x and y axes. With an initial of position of endpoints at $(0, 0)$, $(0, 1)$, it slides continuously to the position $(1, 0)$, $(0, 0)$ sweeping out a region in the positive quadrant. Determine the equation for the boundary of this region.

5. Suppose that $f(x)$ is on continuously differentiable function, with bounded derivative, defined on [0, 1] such that

$$\lim_{x \to 1^-} f(x) - \lim_{x \to 0^+} f(x) = 1.$$

The Fourier coefficients of f are defined by

$$\hat{f}(n) = \int_0^1 f(x) e^{-2\pi i n x} dx.$$

Prove that, as $n \to \pm\infty$, we have:

$$\hat{f}(n) = \frac{i}{2\pi n} + o\left(\frac{1}{n}\right).$$

6. If a real $n \times n$ matrix A has rank m, then show that $A^t A$ also has rank m.

7. Let A be a real $n \times n$ matrix, and define the inner product

$$\langle x, y \rangle = \sum_{j=1}^{n} x_j y_j.$$

We say that A is positive definite if $A = A^t$ and $\langle Ax, x \rangle > 0$ for any $x \neq 0$.

(a) Show that a positive definite matrix is invertible.

(b) Show that if A is positive definite, then so is A^{-1}.

(c) Suppose that A is not symmetric, but $\langle Ax, x \rangle > 0$ for any $x \neq 0$. Is A necessarily invertible?