AMCS Written Preliminary Exam
Part II, April 30, 2015

1. For integers $1 \leq q \leq p$, let

$$a_n = \sum_{k=qn}^{pn+1} \frac{1}{k}. \tag{1}$$

Prove that

$$\lim_{n\to\infty} a_n = \log\left(\frac{p}{q}\right). \tag{2}$$

2. Let $f(z)$ be analytic in the whole complex and suppose that

$$|f(x+iy)| \leq e^x \tag{3}$$

throughout the plane. What can we conclude about $f(z)$?

3. If X and Y are independent random variables, which are uniformly distributed in the interval [0,1], what is Prob($|X - Y| > 1/2$)?

4. Let A be a real 3×3 symmetric matrix with characteristic polynomial

$$\det(tI - A) = t^3 + at + b. \tag{4}$$

Give the characteristic polynomial for A^2, expressing the coefficients in terms of a and b. Hint: Use the fact that if B, C are square matrices of the same size, then $\det(BC) = \det B \cdot \det C$.

5. Find the minimum distance between points on the ellipse

$$\frac{x^2}{4} + y^2 = 1, \tag{5}$$

and the straight line $x + y = 4$.

6. Suppose that A and B are real 2×2 matrices so that $A^k = 0$ for some integer $0 < k$ and $B^l = 0$ for some integer $0 < l$. Is there necessarily an integer $0 < j$ so that $(AB)^j = 0$? You must justify your answer.

7. Consider the series

$$f(z) = \sum_{n=0}^{\infty} \cos(nz) z^n. \tag{6}$$

   (a) Describe the region in which this series converges.
   (b) What is the sum of the series.
   (c) Describe the analytic continuation of the function $f(z)$ defined by the series.