AMCS Written Preliminary Exam
Part II, April 28, 2016

1. Suppose that $\phi : \mathbb{R}^3 \to \mathbb{R}$ is a smooth, strictly positive function satisfying
   - $|\nabla\phi|^2 = 5\phi$.
   - $\Delta(\phi^2) = 15\phi$.

   Evaluate the surface integral

$$\int_{S_1} \frac{\partial\phi}{\partial \boldsymbol{n}} dS \tag{1}$$

   where S_1 is the unit sphere centered at the origin, dS is surface measure on the sphere, and $\frac{\partial\phi}{\partial \boldsymbol{n}}$ is the directional derivative with respect to the unit outward normal vector along S_1.

2. Show that there is a single valued, analytic function defined by the formula

$$f(z) = \sqrt{z^2 - 1},$$

   in the set $\mathbb{C} \setminus [-1, 1]$, which takes positive real values on the ray $(0, \infty)$.

3. A real, $n \times n$ symmetric matrix, A, is positive definite if for any non-zero vector v we have $\langle Av, v\rangle > 0$. Let A and B be real, $n \times n$ symmetric, positive-definite matrices.
   (a) Prove that $aA + bB$ is positive-definite, where a, b are real positive numbers.
   (b) Prove that A^{-1} is symmetric and positive-definite.
   (c) Is AB always symmetric and positive-definite? Give a proof or counterexample.
   (d) Show that the trace of AB is always positive. Hint: If $\{f_1, \dots, f_n\}$ is any orthonormal basis for $\mathbb{R}^n$, then

$$\operatorname{tr}(AB) = \sum_{j=1}^{n} \langle ABf_j, f_j\rangle. \tag{2}$$

4. Show that there is a constant C so that for any 4 real numbers x_1, x_2, x_3, x_4 we have

$$x_1x_2x_3 + x_1x_2x_4 + x_1x_3x_4 + x_2x_3x_4 \leq C(x_1^2 + x_2^2 + x_3^2 + x_4^2)^{\frac{3}{2}}. \tag{3}$$

   what is the smallest value we can take for C?

5. Suppose that X_1 and X_2 are independent random variables, both of which have density $p(x)$ on [0,1], where p is a continuous function. Which of the following statements are always true? Give a brief justification or counterexample in each case.
   (a) $\text{Prob}(X_1 + X_2 < t) < ct^2$ for some constant c and all $t \in [0, 2]$.
   (b) $X_1 + X_2$ has a continuous density g and $g(0) = 0$.
   (c) $E(X_1/X_2)$ is finite.
   (d) $EX_1^{-1/2}$ is finite.
   (e) $\text{Prob}(X_1 - X_2 = t) = 0$ for all t.