AMCS Written Preliminary Exam
Part II, August 29, 2016

1. Suppose that $f(z)$ is a function analytic in the entire complex plane that satisfies

$$|f(z)| \leq C|P(z)| \tag{1}$$

for a constant C and a polynomial $P(z)$.

(a) Show that $f(z)$ is a polynomial.

(b) How is the degree of f related to the degree of P?

(c) How are the zeros of f related to those of P?

2. For $\epsilon > 0$ let

$$c(\epsilon) = \int_{-\infty}^{1-\epsilon} \frac{x}{1-x^3} dx + \int_{1+\epsilon}^{\infty} \frac{x}{1-x^3} dx \tag{2}$$

What is the $\lim_{\epsilon \to 0^+} c(\epsilon)$?

3. Show that the improper integral

$$\int_0^\infty \sin\left(\frac{1}{x}\right) \frac{dx}{x} \tag{3}$$

is convergent, that is:

$$\lim_{R \to \infty} \int_{1/R}^{R} \sin\left(\frac{1}{x}\right) \frac{dx}{x} \tag{4}$$

exists.

4. Find

$$\lim_{n \to \infty} \frac{1}{n^2} \sum_{k=1}^{n} \sqrt{n^2 - k^2}. \tag{5}$$

5. Let A be a real $n \times m$ matrix. The row rank of A is the dimension of the linear space spanned by the rows of A, and the column rank is the dimension of the linear space spanned by the columns of A. Prove that these numbers are always equal.

6. Let X be a finite dimensional real vector space with an inner product $\langle x, y\rangle$. If U is a subspace of X, then we define the subspace

$$U^{\perp} = \{y \in X : \langle u, y\rangle = 0 \text{ for all } u \in U\}. \tag{6}$$

   (a) Show that $X \simeq U \oplus U^{\perp}$.
   (b) Show that $(U^{\perp})^{\perp} = U$.
   (c) Let U and V be 2 subspaces, and let $U + V = \{u + v : u \in U, v \in V\}$. Show that $U + V$ is a subspace and

$$(U + V)^{\perp} = U^{\perp} \cap V^{\perp}. \tag{7}$$

7. Let p_n be the probability that $2n$ independent fair coin flips result in precisely n TAILS. Find a formula for p_n and constants b and c such that

$$\lim_{n\to\infty} p_n/(bn^c) = 1.$$