AMCS Written Preliminary Exam
Part II, August 27, 2015

1. Evaluate the surface integral:

$$\int_S [(xy, y^2, yz) \cdot \boldsymbol{n}(x, y, z)] dA, \tag{1}$$

with $\boldsymbol{n}(x, y, z)$ the outer unit normal vector field. Here $S = \{(x, y, z) : x^2 + y^2 + z^2 = 1\}$. Hint: Use the divergence theorem.
What is the value of this integral over $S_+ = S \cap \{(x, y, z) : \ 0 \leq y\}$?

2. The power series

$$f(z) = \sum_{n=1}^{\infty} nz^n \tag{2}$$

defines an analytic function in the disk $\{z : \ |z| < 1\}$. What is the largest set to which this function has an analytic continuation?

3. Prove or give a counterexample to the following statement: Suppose that X, Y and Z are three real valued random variables, then at least one of the probabilities $P(X \geq Y)$, $P(Y \geq Z)$, or $P(Z \geq X)$ is 1/3 or greater.

4. Suppose that A is an invertible $n \times n$ matrix, and u and v are $n \times 1$ vectors. When is the matrix $A + u \otimes v^t$ invertible? When it is, give a formula for its inverse. The matrix $u \otimes v^t$ acts on a vector by

$$u \otimes v^t \cdot x = \langle x, v \rangle u. \tag{3}$$

Hint: First solve the problem when $A = \text{Id}$.

5. Suppose that $p(x)$ is a real valued, twice continuously differentiable function in $[a, b]$.
   (a) Show that if $p''(x) \geq 0$ in $[a, b]$, then p is convex, that is for $x, y \in (a, b)$ and $\lambda \in (0, 1)$ we have the estimate

$$p(\lambda x + (1 - \lambda)y) \leq \lambda p(x) + (1 - \lambda)p(y). \tag{4}$$

   Hint: Show that we can assume that $p(x) = p(y) = 0$, and then show that $p(\lambda x + (1 - \lambda)y \leq 0$.
   (b) Now show that if p is convex then $0 \leq p''(x)$.
   (c) Suppose that p is defined on $\mathbb{R}$. Show that any local minimum of p is a global minimum. What happens if p assumes it minimum value at two distinct points $x_0 < x_1$?

6. Suppose that two players, X and Y, play the following game: they take turns flipping a coin with probability p of landing on H. The first person to get an H wins. If X goes first, then what is the probability, as a function of p, that X wins? Would you choose to be player X or player Y?