AMCS Written Preliminary Exam
Part II, May 2, 2019

1. Let M be an $n \times n$ matrix with real entries and eigenvalues $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$. Set
$$\rho(M) := \max\{|\lambda_1|, \ldots, |\lambda_n|\}.$$
Show that $\rho(M) < 1$ if and only if $\lim_{k\to\infty} M^k x = 0 \in \mathbb{R}^n$ for every $x \in \mathbb{R}^n$.

2. For $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, set
$$\|x\|_1 := \sum_{i=1}^n |x_i| \quad \text{and} \quad \|x\|_2 := \left(\sum_{i=1}^n x_i^2\right)^{1/2}.$$
Find the largest constant c_1 and smallest constant c_2 such that
$$c_1\|x\|_1 \le \|x\|_2 \le c_2\|x\|_1$$
for each $x \in \mathbb{R}^n$.

3. Prove that
   (a) $\sum_{n=2}^\infty \frac{1}{(\log n)^3}$ diverges.
   (b) $\sum_{n=2}^\infty \frac{1}{(\log n)^{\log n}}$ converges.

4. Let D be the open unit disk in $\mathbb{R}^2$ and suppose that $u \in C(\overline{D}) \cap C^2(D)$ satisfies
$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \lambda u = 0$$
in D for some real number λ and that
$$u|_{\partial D} = 0.$$
Show that $\lambda \ge 0$.

5. (a) What kind of singularity does $\frac{1}{(e^z - 1)}$ have at $z = 0$?
   (b) Find the residue at $z = 0$.

(c) Find the first three non-zero terms of the Laurent series expansion about $z = 0$.

(d) Find the largest number R such that the Laurent series converges on $0 < |z| < R$.

6. Show that all five roots of $z^5 + 15z + 1 = 0$ lie inside the circular disk $|z| < 2$, but that only one of the roots lies inside the disk $|z| < 3/2$.

7. Suppose that $\{X_n\}$ is a sequence of independent discrete random variables such that

$$P(X_n = k) = \frac{e^{-n} n^k}{k!} \text{ for all } k = 0, 1, 2, \ldots.$$

Prove that the expected value of following sum

$$\sum_{i=1}^{n} X_i$$

is $\frac{n(n+1)}{2}$.