Name

AMCS Written Preliminary Exam  
Part II, April 27, 2018

1. Suppose that \( \sum_{n=0}^{\infty} a_n (z - 1)^n \) is the power series expansion of the function 
\[
f(z) = \frac{1}{\cos z}
\]
about the point \( z = 1 \). Does the series
\[
(1) \sum_{n=0}^{\infty} |a_n|
\]
converge or diverge? You must justify your answer.

2. Suppose that \( f(z) \) is an entire function such that \( |f(z)| \leq e^x \), for all \( z = x + iy \). What can be said about the function \( f(z) \)? You must prove your answer.

3. A real matrix \( A \) is skew symmetric if \( A^t = -A \).
   (a) Show that a \( (2n+1) \times (2n+1) \) skew symmetric matrix has a non-trivial null-space.
   (b) Show that if \( \lambda \) is an eigenvalue of \( A \), then so is \(-\lambda\).
   (c) Show that the spectrum of \( A \) is purely imaginary, i.e., consists of numbers of the form \( \{i \lambda_j \} \) where the \( \lambda_j \in \mathbb{R} \).
   (d) Show that \( \det A \geq 0 \).

4. Suppose that \( A \) is a real, upper triangular matrix, with strictly positive diagonal entries. Prove that there is a real, upper triangular matrix, with strictly positive diagonal entries, \( B \), such that \( B^2 = A \). Hint: Use induction.

5. Evaluate the following limits and justify your answers:
   (a) \[
   \lim_{n \to \infty} \int_0^1 n e^{-nx} (\cos x)^2 \, dx.
   \]
   (b) \[
   \lim_{t \to \infty} \left[ te^t \int_t^\infty \frac{e^{-s}}{s} \, ds \right].
   \]

6. Find a conformal map from the half disk 
\[
D_1^+ = \{ z : |z| < 1 \text{ and } 0 < \text{Im } z \}
\]
to the unit disk, \( D_1 = \{ z : |z| < 1 \} \).
Two random points $A$ and $B$ are selected independently, and uniformly from the disk \{(x, y) : x^2 + y^2 < 1\}. A third random point $C$ is selected uniformly from the larger disk \{(x, y) : x^2 + y^2 < 4\}, independently of $A$ and $B$. What is the probability that the angle $\angle ACB$ is obtuse? Hint: First consider the answer for a fixed choice of $A$, $B$. 