Name

AMCS Written Preliminary Exam
Part II, April 27, 2018

1. Suppose that $\sum_{n=0}^{\infty} a_n(z-1)^n$ is the power series expansion of the function
$$f(z) = \frac{1}{\cos z}$$
about the point $z = 1$. Does the series
$$\sum_{n=0}^{\infty} |a_n| \tag{1}$$
converge or diverge? You must justify your answer.
2. Suppose that $f(z)$ is an entire function such that $|f(z)| \leq e^x$, for all $z = x + iy$. What can be said about the function $f(z)$? you must prove your answer.
3. A real matrix A is skew symmetric if $A^t = -A$.
   (a) Show that a $(2n+1)\times(2n+1)$ skew symmetric matrix has a non-trivial null-space.
   (b) Show that if λ is an eigenvalue of A, then so is $-\lambda$.
   (c) Show that the spectrum of A is purely imaginary, i.e., consists of numbers of the form $\{i\lambda_j\}$ where the $\lambda_j \in \mathbb{R}$.
   (d) Show that $\det A \geq 0$.

4. Suppose that A is a real, upper triangular matrix, with strictly positive diagonal entries. Prove that there is a real, upper triangular matrix, with strictly positive diagonal entries, B, such that $B^2 = A$. Hint: Use induction.
5. Evaluate the following limits and justify your answers:
   (a)
$$\lim_{n\to\infty} \int_0^1 ne^{-nx}(\cos x)^2 dx.$$
   (b)
$$\lim_{t\to\infty} \left[te^t \int_t^{\infty} \frac{e^{-s}}{s} ds \right].$$
6. Find a conformal map from the half disk
$$D_1^+ = \{z : |z| < 1 \text{ and } 0 < \operatorname{Im} z\}$$
to the unit disk, $D_1 = \{z : |z| < 1\}$.

7. Two random points A and B are selected independently, and uniformly from the disk $\{(x, y) : x^2 + y^2 < 1\}$. A third random point C is selected uniformly from the larger disk $\{(x, y) : x^2 + y^2 < 4\}$, independently of A and B. What is the probability that the angle $\angle ACB$ is obtuse? Hint: First consider the answer for a fixed choice of A, B.