AMCS Written Preliminary Exam
Part II, May 1, 2017

1. There are only a few entire functions that satisfy the relation
$$f(f(z)) = f(z).$$
Find them all and justify your answer.

2. Evaluate the following integrals
$$\int_0^\infty \frac{x^\alpha dx}{x^2+1} \text{ and } \int_0^\infty \frac{x^\alpha \ln x dx}{x^2+1},$$
for $0 < \alpha < 1$. On $(0, \infty)$ the function x^α is defined to be real and positive, and $\ln x$ is defined to be real.

3. Show that the improper integral
$$\int_0^\infty \sin\left(x^2\right) dx$$
is convergent, that is:
$$\lim_{R\to\infty} \int_0^R \sin\left(x^2\right) dx$$
exists. Extra Credit: Compute the value of this integral.

4. For the following series state whether the series is absolutely convergent, conditionally convergent, or divergent. You must justify your answers by stating which theorems or tests you have applied

(a)
$$\sum_{n=1}^\infty \frac{\log\left(1+\frac{1}{n}\right)}{n}$$

(b)
$$\sum_{n=1}^\infty \left(1+\frac{1}{n^2}\right)^{n^2}$$

(c)
$$\sum_{n=1}^\infty \frac{(-1)^n}{\sqrt{n}}$$

5. Let A be a real $n \times n$ matrix with entries (a_{ij}). We define the trace of A to be
$$\operatorname{tr}(A) = \sum_{i=1}^{n} a_{ii}.$$
   (a) Show that if B is another $n \times n$ matrix, then $\operatorname{tr}(BA) = \operatorname{tr}(AB)$.
   (b) If $\{\lambda_1, \ldots, \lambda_n\}$ are the eigenvalues of A, repeated according to their multiplicity, then
$$\operatorname{tr}(A) = \sum_{i=1}^{n} \lambda_i.$$
   Be careful: not every matrix is diagonalizable!
   (c) For which n does there exist a pair of $n \times n$ matrices A, B so that
$$AB - BA = \mathrm{Id}\,.$$
   You must prove your answer.
6. Let X be a finite dimensional vector space, with inner product $\langle \cdot, \cdot \rangle$, and let U, V be subspaces of X. For a subspace U we let $U^\perp$ denote its orthogonal complement.
   (a) Prove that $\dim(U + V) = \dim U + \dim V - \dim U \cap V$.
   (b) Prove that $(U + V)^\perp = U^\perp \cap V^\perp$.
   (c) Let P_U and P_V be the orthogonal projections onto U and V respectively. When is it true that $P_U P_V = P_V P_U$?
7. A high-risk, high-gain investment opportunity will either triple or halve your investment by the end of a year, with respective probabilities $1/3$ and $2/3$. The performance each year is independent of other years. Let X_n be the amount of your fortune after n years, if your initial fortune $X_0 = 1$. We define the random variable
$$Y = \limsup_{n \to \infty} X_n$$
   With probability 1, Y takes the value 0, or infinity. Decide which and prove your answer. Hint: consider $\log Y$.