# AMCS Written Preliminary Exam
# Part II, August 27, 2018

1. Let $V$ be the vector space of $C^\infty$ functions $y$ on the interval $(-1,1)$ which satisfy the ODE $y''' - 2y'' + y' = 0$. Show that the derivative operator $\frac{d}{dx}$ maps $V$ to itself and find a basis of $V$ such that the matrix of $\frac{d}{dx}$ in this basis is in Jordan canonical form.

2. Find the eigenvalues and eigenvectors of the following matrix $A$:

$$A = \begin{pmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{pmatrix}$$

   What is the $\lim_{k\to\infty} A^k = ?$

3. Suppose that $\vec{F}$ is the following vector field:

$$\vec{F}(x,y,z) = \left\langle \frac{1}{3}x^3 - 3xy^2, \frac{1}{3}y^3 - 3yz^2, \frac{1}{3}z^3 - 3zx^2 \right\rangle.$$

   Compute the flux integral

$$\int_S \vec{F} \cdot \vec{n}\, dA,$$

   where $\vec{n}$ is the outward normal to the surface

$$S := \left\{(x,y,z) \ : \ x^2 + y^2 + z^2 = 1\right\}.$$

4. Show that there exists a real-valued function $f$ on the interval $(-1,1)$ which is differentiable at every point with a derivative $f'$ which is not continuous.

5. Compute the Laurent series in the annulus $1 < |z| < 2$ of the function

$$\frac{1}{z^8 - 17z^4 + 16}.$$

6. Let $z_n$ be the real zero of the polynomial

$$1 - \frac{1}{2!}\left(\frac{z}{2}\right)^2 + \frac{1}{4!}\left(\frac{z}{2}\right)^4 - \cdots + \frac{(-1)^n}{(2n)!}\left(\frac{z}{2}\right)^{2n}$$

which is closest to $\pi$.

(a) Prove that this sequence of zeroes denoted $\{z_n\}$ converges as $z_n \to \pi$ as $n \to \infty$.

(b) Show that $R^n|z_n - \pi| \to 0$ as $n \to \infty$ for any positive real number $R$.

You may solve this for example by using the mean value theorem.

7. A fair coin is tossed repeatedly until it lands on tails for the $k$-th time, at which point the game ends. During the game, a sequence of repeated, consecutive heads is called a winning streak. Show that there is probability greater than $1 - e^{-1}$ that the longest winning streak is at least $\log_2 k$.