AMCS Written Preliminary Exam
Part II, August 28, 2017

1. Find a 1-to-1 conformal map from the sector $S_{\frac{\pi}{4}} = \{z : 0 < \arg z < \frac{\pi}{4}\}$ onto the interior of the unit disk $D_1 = \{z : |z| < 1\}$.

2. Let $< f_n >$ be a sequence of functions analytic in the unit disk, D_1, and let f be a continuous function also defined in D_1. Show that if

   (1)
   $$\lim_{n\to\infty} \iint_{D_1} |f_n(x, y) - f(x, y)| dx dy = 0,$$

   then the function f is also analytic in D_1.

3. Suppose that A is an $n \times n$ matrix that commutes with all $n \times n$ diagonal matrices. What can we say about A; you must prove your answer.

4. (a) What is a Hermitian inner product?
   (b) What is unitary matrix?
   (c) What is a Hermitian matrix?
   (d) Suppose that A is an Hermitian matrix. Prove that $(A - i \text{ Id})$ is invertible and the matrix $Z = (A + i \text{ Id})(A - i \text{ Id})^{-1}$ is a unitary matrix.

5. Show that there is no non-constant polynomial $P(u, v)$ in two variables such that

   (2)
   $$P(x, \cos x) = 0$$

   holds for all $x \in \mathbb{R}$.

6. Consider the equation $ye^y = x$. Show that this uniquely defines a function $y(x)$ on the interval $[0, \infty)$.
   (a) Sketch the graph of $y(x)$.
   (b) Find a formula for $\frac{dy}{dx}$ as a function of y.
   (c) Give a method to calculate the value of y such that $ye^y = 1$, to arbitrary accuracy.

7. Fix a large number N and for $1 \leq j \leq N$ let $\{U_j\}$ be IID random variables uniform on the interval $[0, 1]$. Let X be the number of these that are local maxima, that is, the number of j such that $U_{j-1} < U_j > U_{j+1}$. The indices are considered modulo N, e.g., a maximum occurs at 1 if $U_N < U_1 > U_2$.

   (a) Compute the expected value $\mu := E(X)$.

   (b) Compute the second moment $M := E(X^2)$.

   (c) Compute the variance $V := \text{Var}(X)$.

   (d) For what real number q can you prove a nontrivial upper bound on

   $$\text{Prob}(|X - \mu| > cN^q)$$

   that does not depend on N? It will of course depend on c, and nontrivial means it has to be less than 1 for at least a range of values of c. You are not asked to get the best bound, just any bound uniform in N.

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