AMCS Written Preliminary Exam
Part II, August 28, 2013

1. Compute the inverse of the following matrix

$$\begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 2 \\ 2 & 0 & 0 & 1 \end{bmatrix}.$$

2. Consider the following sequence:

$$0, \frac{1}{3}, \frac{2}{3}, 0, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, 0, \frac{1}{7}, \frac{2}{7}, \frac{3}{7}, \frac{4}{7}, \frac{5}{7}, \frac{6}{7}, 0, \frac{1}{9}, \ldots$$

   For a natural number N, consider the first N terms of this sequence, and let I_N be the number of such terms which happen to fall within the interval $[a, b] \subset (0, 1)$. Show

$$\lim_{N\to\infty} \frac{I_N}{N} = b - a.$$

3. Show that

$$\int_0^\infty \frac{\log x\, dx}{x^2 + a^2} = \frac{\pi \log a}{2a} \text{ when } a > 0.$$

4. Find a one-to-one conformal map from the strip

$$S = \{z : \ 0 < \operatorname{Im} z < \pi\}$$

   to the unit disk $D_1 = \{z : \ |z| < 1\}$.

5. Let p_n be the probability that $2n$ independent fair coin flips result in precisely n HEADS. Find the constants b and c such that

$$\lim_{n\to\infty} \frac{p_n}{bn^c} = 1.$$

6. Prove that

$$\lim_{N\to\infty} \sum_{k=0}^{\infty} \left(1 + \frac{k}{N}\right)^{-N} = \frac{e}{e-1}.$$

   Any interchange of limits **must** be carefully justified.

7. Suppose that A is a symmetric 3×3 matrix with positive entries, such that the sum of every row is 1. Let $v = (1, \frac{1}{2}, \frac{1}{3})$. What is

$$\lim_{n\to\infty} A^n v?$$