AMCS Written Preliminary Exam
Part II, August 28, 2012

1. Show that the radius of convergence of the power series

$$\sum_{n=1}^{\infty} \frac{(-1)^n z^{n(2n+1)}}{n}, \tag{1}$$

is 1, and discuss convergence at the points $z = 1, -1$ and i.

2. Find a conformal map from

$$D = \left\{ z : \ |z| < 1 \text{ and } 0 < \arg z < \frac{\pi}{2} \right\} \tag{2}$$

to the upper half plane $H_+ = \{z : \ \operatorname{Im} z > 0\}$.

3. Discuss the convergence properties of the following series:

$$\begin{aligned} &\sum_{n=0}^{\infty} z^n \\ &\sum_{n=1}^{\infty} \frac{z^n}{n^\alpha} \quad \text{for } \alpha > 0 \\ &\sum_{n=1}^{\infty} n^\alpha z^n \quad \text{for } \alpha > 0. \end{aligned} \tag{3}$$

4. Let A_n be a sequence of square matrices converging to A. Give a proof or counterexample:
   (a) If each A_n is singular, then A is singular.
   (b) If each A_n is non-singular, then A is non-singular.

5. Let A and B be real symmetric matrices, with spectra:

$$\begin{aligned} &\lambda_1(A) \leq \cdots \leq \lambda_n(A) \\ &\lambda_1(B) \leq \cdots \leq \lambda_n(B). \end{aligned} \tag{4}$$

Show that if $\langle Ax, x \rangle < \langle Bx, x \rangle$ for all non-zero $x \in \mathbb{R}^n$, then $\lambda_1(A) < \lambda_1(B)$, and $\lambda_n(A) < \lambda_n(B)$.

6. Suppose that $\{u_1, \dots, u_n\}$ and $\{v_1, \dots, v_n\}$ are orthonormal bases. Define the $n \times n$ matrix

$$A = \sum_{j=1}^{n} \mu_j v_j \otimes u_j^t, \tag{5}$$

where $\{\mu_j\}$ are complex numbers. Recall that $u \otimes v^t x \overset{d}{=} \langle x, v \rangle u$. What is determinant A?

7. Suppose that $\varphi(x)$ is a smooth function of compact support on $\mathbb{R}$ and, if $\operatorname{Im} z \neq 0$, define

$$f(z) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\varphi(\xi) d\xi}{(\xi - z)}. \tag{6}$$

Prove that, for each $x \in \mathbb{R}$, the limit

$$\lim_{\epsilon \to 0^+} [f(x + i\epsilon) - f(x - i\epsilon)] \tag{7}$$

exists, and compute what it is.

8. Suppose that X_1 and X_2 are independent random variables, uniformly distributed in $[-1, 1]$. What is the probability that (X_1, X_2) lies inside the unit circle centered at $(0, 0)$.