AMCS Written Preliminary Exam, II
August 25, 2009

All work should go in the exam booklet, with your final answer clearly marked.

1. Let V be the real vector space of real polynomials of degree at most 5. We define the map $T : V \to \mathbb{R}^3$ by $T(p) = (p(0), p(1), p(2))$.
   (a) Show that T is a linear transformation.
   (b) Find the dimensions of the kernel and image of T, and find a basis for each.

2. Let $f(z)$ be an entire function which is real on the real axis, with $f(0) = 1$. Assume that there is a real constant M such that $|f(z)| \leq M$ for all z satisfying $\text{Im}(z) \geq 0$. Evaluate
$$\lim_{x\to\infty} f(x).$$
   Your must prove your answer.

3. Compute (in the complex plane)
$$\int_{|z|=2} \frac{dz}{z^2+1}$$

4. **Use the fact that** $z \mapsto z^3$ **is a conformal map** to prove that for any $a, b \in \mathbb{R}$, not both zero, the plane curves
$$C_a = \{(x, y) : \; x^3 - 3xy^2 = a\} \text{ and } D_b = \{(x, y) : \; 3x^2y - y^3 = b\}$$
   intersect at three distinct points, and, at these points of intersection, the tangent vectors to these curves are perpendicular.

5. The faces of a six-sided die are marked 1, 2, 2, 3, 3, 4. The faces of another six-sided die are marked by six natural numbers, call them a, b, c, d, e, f. The total, when both of these dice are rolled, has the same distribution as the total on a pair of ordinary six-sided dice, both marked 1, 2, 3, 4, 5, 6. Find the set of numbers $\{a, b, c, d, e, f\}$ for which this is true.

6. If X is a random variable uniformly distributed on the interval [0, 1], what is the expected value of $\log X$? (Here log is the natural logarithm).

7. An integer valued random variable X is a Poisson random variable of strength $\lambda > 0$ if
$$\text{Prob}(X = k) = \frac{\lambda^k e^{-\lambda}}{k!}.$$
Suppose that Y is another Poisson random variable of strength $\mu > 0$. If X and Y are independent random variables, show that $X + Y$ is also a Poisson random variable of strength $\lambda + \mu$. Hint: Consider the expectation of $s^{(}X + Y)$.