AMCS Written Preliminary Exam
Part II, August 26, 2008

1. Prove that there exists a constant $C \in \mathbb{R}$ so that
$$\sum_{k=1}^{N} \frac{1}{k} = \log N + C + O\left(\frac{1}{N}\right).$$

2. For each $n \in \mathbb{N}$, show that there is a polynomial $T_n(x)$ so that
$$T_n(\cos\theta) = \cos n\theta.$$
Prove that for $m \neq n$
$$\int_{-1}^{1} \frac{T_n(x)T_m(x)dx}{\sqrt{1-x^2}} = 0.$$

3. Suppose that f is an analytic function defined in $D_1 \subset \mathbb{C}$, which defines a 1-1 map onto U. Show that
$$\text{Area}(U) = \iint_{D_1} |f'(z)|^2 dx dy.$$

4. True or false. Give your answer with a brief explanation. Here $f(z)$ is an entire function.
   (a) If $f(x) = 0$ for all real x in the interval $(0, 1)$, then $f(z) \equiv 0$, for all complex z.
   (b) If $f(x)$ is bounded for all real x, then f is a constant function.
   (c) If $|f(z)| \to \infty$ as $|z| \to \infty$, then f is a polynomial.
   (d) $|e^z| \leq e^{|z|}$ for all $z \in \mathbb{C}$.

5. Find the Jordan canonical form of
$$A = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 0 & 2 \\ 0 & -1 & 3 \end{bmatrix}.$$

6. Suppose that A is a real symmetric $n \times n$ matrix. Prove that all the eigenvalues of A are real, and that A has an orthonormal basis of eigenvectors.

7. Suppose that a data set $\{x_n : n = 1, \ldots, N\}$, with $N = 500,000$ has average

$$\overline{x} = \frac{1}{N} \sum_{n=1}^{N} x_n = 13.06,$$

and root mean square

$$\sigma = \sqrt{\frac{1}{N} \sum_{n=1}^{N} x_n^2} = 13.67.$$

Using this information, derive the best upper bound that you can, for the number of measurements that are greater than 17.1.