AMCS Written Preliminary Exam
Part I, April 30, 2015

1. Let x_0 be a positive real number; we define a sequence of real numbers by setting:
$$x_{n+1} = x_n + \frac{16 - x_n^4}{4x_n^3}.$$
Show that the sequence converges to 2.

2. Show that
$$\cos x \geq 1 - \frac{x^2}{2} \tag{1}$$

3. Evaluate the contour integral
$$I = \int_0^{2\pi} \frac{d\theta}{4 + \cos\theta}. \tag{2}$$

4. What are the singularities of the function
$$f(z) = \frac{ze^{\frac{1}{(1-z)^2}}}{\sin \pi z}? \tag{3}$$
Label each as a pole, removable singularity, or essential singularity. Remember the "point at infinity."

5. Let $(x(y), y(t))$ be a solution to the system of ordinary differential equations
$$\begin{aligned} \dot{x}(t) &= -x(t) - y(t) \\ \dot{y}(t) &= -y(t) + x(t). \end{aligned} \tag{4}$$
Prove that $\lim_{t\to\infty}(x(t), y(t)) = (0, 0)$.

6. For what values of a is the following matrix positive definite
$$A = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 5 & a \\ 1 & a & 2 \end{pmatrix}?$$

7. Suppose that we flip a fair coin n times. Show that
   (a) For any $0 < \epsilon$, show that the expected value of $(1 + \epsilon)^{\#\text{TAILS}}$ equals $(1 + \frac{\epsilon}{2})^n$.
   (b) Use Chebyshev's inequality to conclude that the probability of producing more than $\frac{2n}{3}$ TAILS tends to zero exponentially as $n \to \infty$.