AMCS Written Preliminary Exam
Part I, April 28, 2016

1. Show that

$$\sum_{n=2}^{\infty} \frac{1}{(\log n)^{\log n}} \tag{1}$$

is a convergent sum, whereas

$$\sum_{n=2}^{\infty} \frac{1}{n \log n \log \log n} \tag{2}$$

is divergent.

2. Show that the following limit exists

$$\lim_{N\to\infty} \sum_{j=1}^{N} \frac{(-1)^{j+1}}{j}. \tag{3}$$

Prove that this limit equals $\log 2$.

3. Evaluate the following limit

$$\lim_{t\to\infty} \int_{c-it}^{c+it} \frac{e^{az}dz}{z^2}. \tag{4}$$

Here a is a real number and c is a positive real number. Note that you need to carefully consider the consequences of the sign of a.

4. Map the semi-circle $D_+ = \{(x, y) : x^2 + y^2 < 1,\ 0 \leq y\}$ conformally onto the upper half plane. Choose the map that carries $z = -1$ to 0.

5. Consider the following systems of linear equations:

$$\begin{pmatrix} \lambda & 1 & 1 \\ 1 & \lambda & 1 \\ 1 & 1 & \lambda \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ \mu \end{pmatrix}. \tag{5}$$

   (a) Determine all pairs (λ, μ) for which the equation has a unique solution.
   (b) Determine all pairs (λ, μ) for which the equation has no solution.

6. For each $n = 1, 2, 3, \ldots$ show that there is a unique polynomial $b_n(x)$ that satisfies the equation

$$\int_{x}^{x+1} b_n(t)dt = x^n. \tag{6}$$

Find $b_1(x)$, and $b_2(x)$. Hint: Existence and uniqueness are proved used different arguments.

7. A stick of length 1 is broken into three pieces “randomly.” Give a precise definition for this and, with your definition, compute the probability that these pieces are the edges of a triangle.