AMCS Written Preliminary Exam
Part I, August 29, 2016

1. For which values of x is the following series convergent

(1)
$$\sum_{n=1}^{\infty} \frac{\cos(nx)}{n}?$$

How about

(2)
$$\sum_{n=1}^{\infty} \frac{\cos(nx)}{n^{\frac{1}{4}}}?$$

2. Let $f(z)$ be analytic in $|z| < 1$ and suppose that $f(0) = 0$. Prove that

(3)
$$G(z) = \sum_{n=1}^{\infty} f(z^n)$$

is also analytic in $|z| < 1$. You can use the fact that a uniformly convergent sequence of analytic functions has an analytic limit. If the limit $G(z)$ is constant, then what can you say about f? You must justify your answer.

3. Let V be an $(n+1)$-dimensional real vector space of real valued functions defined on the interval $[0, 1]$. Show that if $0 \leq x_1 < \cdots < x_n \leq 1$ are arbitrary, then there is a function $f \in V$ such that

(4)
$$f(x_i) = 0 \text{ for } i = 1, \ldots, n.$$

4. Suppose that (a_{ij}) is a positive definite, symmetric $n \times n$ matrix, that is

(5)
$$A(u) = \sum_{i,j=1}^{n} a_{ij} u_i u_j \geq 0,$$

and $A(u) = 0$ only if $u = 0$. Suppose that $(f_1, \ldots, f_n) \neq (0, \ldots, 0)$ is real vector. Show that the function

(6)
$$G(u) = A(u) + \sum_{j=1}^{n} f_j u_j$$

attains its minimum value. Show that this minimum value is negative and find equations to determine the vector u where this value is attained. What is the minimum value?

5. Let $\mathcal{P}_n$ denote the vector space of polynomials of degree at most n. Prove that, for each n, the differential operator

$$L = x(1-x)\partial_x^2 + \frac{1}{2}x\partial_x \tag{7}$$

maps $\mathcal{P}_n$ to $\mathcal{P}_n$, Show that, for each n, $L : \mathcal{P}_n \to \mathcal{P}_n$ is diagonalizable and find its spectrum. Hint: Consider the operator $-x^2\partial_x^2 + \frac{1}{2}x\partial_x$.

6. Suppose that two buses arrive independently at a bus stop so that, for $0 \leq t$, the arrival times satisfy

$$\text{Prob}(t \leq X < t + dt) = \lambda e^{-\lambda t} dt \text{ and } \text{Prob}(t \leq Y < t + dt) = \mu e^{-\mu t} dt, \tag{8}$$

where λ and μ are positive constants. What is the

$$\text{Prob}(\min\{X, Y\}) < T?$$

7. A fair coin is flipped repeatedly. What is the expected value for the number of flips needed to first see two HEADS in a row? Hint: Write formulas for the expected values conditional on the outcome of the last flip.