AMCS Written Preliminary Exam
Part I, August 26, 2015

1. Let $\{x_n\}$ be a monotonely decreasing sequence of positive numbers, which converges to zero. Show that the infinite sum

$$\sum_{n=1}^{\infty} x_n e^{\frac{\pi i n}{2}} \tag{1}$$

converges.

2. Let $0 < \alpha$. Show that there are positive constants C_0, C_1 so that, for x and y non-negative we have the inequalities:

$$C_0(x+y)^\alpha \le (x^\alpha + y^\alpha) \le C_1(x+y)^\alpha \tag{2}$$

3. Use contour integration to evaluate the following integral:

$$\int_0^\infty \frac{\sqrt{x}dx}{1+x^2}. \tag{3}$$

4. What is the radius of convergence of the power series about $z = 0$ for the function

$$f(z) = \frac{z^2-3}{(4+z)(e^z+1)}?$$

What sort of singularity (or singularities) does it have on the boundary of the disk where it converges?

5. Suppose that A is a strictly upper triangular $n \times n$ matrix, i.e. $A_{ij} = 0$ if $j \le i$. Show that $\exp(tA)$ is a polynomial of degree n in the variable t with matrix coefficients. For the exponential of a matrix B use the power series formula:

$$\exp(B) = \sum_{j=1}^{\infty} \frac{B^j}{j!} \tag{4}$$

Show that the vector equation $\partial_t X(t) = AX(t)$ always has a non-zero solution that is bounded for all $t \in \mathbb{R}$. Here $X(t)$ is an $n \times 1$-vector valued function.

6. Let A and B be matrices that do not commute, that is $AB \neq BA$. Assuming that $A + B$ is invertible, show that

$$A(A+B)^{-1}B = B(A+B)^{-1}A. \tag{5}$$

Hint: One way to do this is to first assume that A is invertible, and then use continuity, but there are other ways.

7. Suppose that we have 3 containers each holding 2 balls. The first container holds 2 red balls, the second container, a red and a black ball, and the third container 2 black balls. A container is selected at random (all containers are equally likely), and from that container a ball is selected at random. If this ball is black, then what is the probability that the other ball in the chosen container is black? You must explain your answer.