AMCS Written Preliminary Exam
Part I, May 2, 2019

1. Calculate the Jordan form of the following matrix:

$$B = \begin{pmatrix} 4 & 2 & 1 \\ -2 & 0 & -1 \\ -1 & -1 & 1 \end{pmatrix}$$

   For this problem you may assume that the following equation holds:

$$\det(B - \lambda I) = (\lambda - 2)^2(\lambda - 1)$$

   (a) Find the algebraic and geometric multiplicities of both eigenvalues.

   (b) Find the Jordan form J of B.

   (c) Find an invertible matrix S so that $B = SJS^{-1}$. Be sure to clearly explain how you obtained the columns of this matrix.

2. Assume A is a symmetric $n \times n$ matrix with eigenvalues $1, 2, \dots, n$ and consider $y : [0, \infty) \to \mathbb{R}^n$ which satisfies the ODE

$$\dot{y}(t) + Ay(t) = 0, \quad t > 0.$$

   (a) Compute the limit

$$\lim_{t\to\infty} e^t y(t).$$

   (b) Provided $y(0) \neq 0$, show $y(t) \neq 0$ for all $t > 0$ and compute

$$\lim_{t\to\infty} \frac{Ay(t) \cdot y(t)}{\|y(t)\|^2}.$$

3. Suppose $f : \mathbb{R}^n \to \mathbb{R}$ satisfies

$$f(tx) = tf(x)$$

   for each $x \in \mathbb{R}^n$ and $t \geq 0$.

(a) Assume that f is differentiable away from 0, then verify that

$$\nabla f(x) \cdot x = f(x)$$

holds away from 0.

(b) Show that if f is differentiable at $x = 0$, then there is $a \in \mathbb{R}^n$ such that $f(x) = a \cdot x$ for all $x \in \mathbb{R}^n$.

4. Prove the summation by parts formula:

$$\sum_{n=1}^{N} a_n b_n = a_N B_N - a_1 B_0 - \sum_{n=1}^{N-1} (a_{n+1} - a_n) B_n,$$

where $\{a_n\}_{n=1}^N$ and $\{b_n\}_{n=1}^N$ are any finite sequences of complex numbers and $\{B_n\}_{n=0}^N$ is any sequence satisfying $B_n - B_{n-1} = b_n$ for any $n = 1, \ldots, N$.

5. Identify all poles, zeros, removable singularities, and essential singularities of the function

$$f(z) = \frac{z^4 + 1}{z^2 - z} \sin \frac{\pi z^2}{z^2 - 1}.$$

Be sure to consider the point at infinity.

6. Show that $\displaystyle\int_0^\infty \frac{x^\beta}{(x+1)^2} dx = \frac{\pi\beta}{\sin \pi\beta}$ assuming that $0 < \beta < 1$.

7. Suppose U, V are two independent random variables which are uniformly distributed in the interval $(0, 1)$. Find the joint distribution of the random variables X, Y defined via

$$X := \sqrt{-2 \ln U} \cos(2\pi V) \quad \text{and} \quad Y := \sqrt{-2 \ln U} \sin(2\pi V).$$