Name

AMCS Written Preliminary Exam
Part I, April 27, 2018

1. Prove that the equation $x = e^{-x}$ has exactly one root in the interval $0 \leq x \leq 1$.
2. Describe as completely as you can the set of all 2×2-matrices, A, that satisfy the equation $A^2 = A$.
3. The second order Taylor series for $\sqrt{25 + x}$ is given by

$$T_2(x) = 5 + \frac{x}{10} - \frac{x^2}{1000}. \tag{1}$$

   Give the best estimate you can for $|\sqrt{24} - T_2(-1)|$. Is the error,

$$\sqrt{24} - T_2(-1),$$

   positive or negative?
4. Suppose that $f(z)$ is an analytic function in $\{z : |z| \leq 1\}$. Find the domain of analyticity of the function

$$F(z) = \frac{1}{2\pi i} \oint_C \frac{f(\zeta) d\zeta}{\zeta^2 - z^2}, \tag{2}$$

   where $C = \{\zeta : |\zeta| = 1\}$. Compute $F(z)$ throughout its domain of analyticity.
5. Let f be a $\mathcal{C}^2$-function defined on $\mathbb{R}$ such that $f''(x) \geq 0$ for every x. Show that if f assumes its maximum value, then f is constant.
6. Let A and B be complex $n \times n$ matrices. Show that AB and BA have the same characteristic polynomials. Hint: This is easier if one of the matrices is invertible.
   How is the spectrum of AB related to the spectrum of BA?
7. Let $\{X_n\}$ be IID geometric random variables with mean 4, that is, for $k \in \{1, 2, \ldots, \}$,

$$\text{Prob}(X_n = k) = \left(\frac{1}{4}\right)\left(\frac{3}{4}\right)^{k-1}. \tag{3}$$

   Let $S_n = \sum_{j=1}^n X_j$ denote the partial sums. Given a positive integer n, what is the probability of the event that $S_n = 10$? What is the probability of the event: "For some $n \in \mathbb{N}$, $S_n = 10$."?