AMCS Written Preliminary Exam
Part I, May 1, 2017

1. Let $f(x)$ be a non-negative, monotone decreasing function for which the integral
$$\int_0^\infty f(x)dx < \infty.$$
Prove that
$$\lim_{x\to\infty} xf(x) = 0.$$

2. Find three different Laurent expansion for function
$$f(z) = \frac{1}{1+3z+z^2}.$$
State where each expansion is valid.

3. Let A be the 9×9 matrix
$$A = \begin{pmatrix} 2s & t & t & t & \cdots & t \\ t & 2s & t & t & \cdots & t \\ t & t & 2s & t & \cdots & t \\ \cdots & \cdots & & & \cdots & t \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ t & t & t & t & \cdots & 2s \end{pmatrix}$$
(All off-diagonal entries are t and all diagonal entries are $2s$.) For which complex values of t and s is this matrix invertible?

4. Find the maximum of $x_1^2 \cdot x_2^2 \cdots x_n^2$ subject to the constraint
$$x_1^2 + \cdots + x_n^2 = 1.$$
From the solution to this problem prove that, for arbitrary positive real numbers $r_1, \ldots, r_n$, we have the inequality
$$(r_1 \cdots r_n)^{\frac{1}{n}} \le \frac{r_1 + \cdots + r_n}{n}.$$
Prove this directly, without recourse to induction.

5. Suppose that A is an invertible $n \times n$ matrix with characteristic polynomial

$$\det(A - \lambda\,\mathrm{Id}) = \sum_{j=0}^{n} a_j \lambda^j.$$

What is the characteristic polynomial of A^{-1}? If B is a 2×2 matrix with characteristic polynomial

$$\det(B - \lambda\,\mathrm{Id}) = \lambda^2 - 3\lambda - 3,$$

then what is the characteristic polynomial of $2B - 3\,\mathrm{Id}$?

6. Let X and Y be independent, standard normal, random variables, that is, their joint density is

$$p(x, y) = \frac{1}{2\pi} e^{-\frac{x^2+y^2}{2}}.$$

Give an expression for $\mathrm{Prob}(X + Y < -1)$ in terms of the normal CDF

$$\Phi(x) := \int_{-\infty}^{x} \frac{e^{-\frac{t^2}{2}}\,dt}{\sqrt{2\pi}}.$$

7. A simple random walk is a particle moving, at each time step, either left or right by 1 unit, each with probability 1/2. What is the average amount of time it takes a simple random walk, started at 1, to hit the set $\{0, 5\}$?