AMCS Written Preliminary Exam
Part I, August 27, 2018

1. A subspace V of $\mathbb{R}^3$ is spanned by the columns of

$$A = \begin{pmatrix} 1 & 1 \\ -1 & 0 \\ 1 & 1 \end{pmatrix}$$

   (a) Apply the Gram-Schmidt process to find two orthonormal vectors u_1 and u_2 which also span V.

   (b) Find an orthogonal matrix Q so that QQ^T is the matrix which orthogonally projects vectors onto V.

   (c) Find the best possible (i.e., least squared error) solution to the linear system

$$Q \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$$

2. Find the Singular Value Decomposition (SVD) of the matrix:

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 1 \end{pmatrix}$$

   (a) Find the eigenvalues and unit length eigenvectors for A^TA and AA^T. (What is the rank of A?)

   (b) Calculate the three matricies U, D, and V^T in the SVD and observe that $A = UDV^T$. Please explain clearly how you obtain these matrices.

3. (a) For any two positive real numbers a and b, prove that

$$1 + \ln a - \ln b \le \frac{a}{b}.$$

   (b) Assume that $\{x_n\}$ is a sequence of nonzero complex numbers satisfying that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \left| \frac{x_{n+1}}{x_n} \right| = r < 1.$$

   Show that the series $\sum_{n=1}^{\infty} x_n$ must be absolutely convergent.

4. Suppose $\{a_n\}$ is a sequence of real numbers such that $\sum_{n=1}^{\infty} |a_n| < \infty$. Show that

$$\lim_{r\to 1^-} \sum_{n=1}^{\infty} a_n r^n = \sum_{n=1}^{\infty} a_n.$$

5. Find all analytic functions on the disk $\{z \in \mathbb{C} \,:\, |z| < 1\}$ which satisfy the functional equation

$$f(iz) = (f(z))^2.$$

6. Calculate the following integral

$$\int_{|z|=\rho} \frac{z+1}{z^2 - 2z}\, dz,$$

   both when $0 < \rho < 1$ and when $1 < \rho < \infty$.

7. Suppose $\{x_n\}$ is a sequence of real numbers in $[0, 1]$ which is randomly constructed according to the rule that given the values of x_{n-1}, x_n is uniformly distributed on the interval $[0, x_{n-1}]$. Compute the expected sum of the infinite series

$$\sum_{n=1}^{\infty} x_n.$$

   You may assume that this sum is finite with probability 1.