AMCS Written Preliminary Exam
Part I, August 28, 2017

1. Prove that

$$\log 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots \tag{1}$$

2. Suppose that $f(z)$ is an analytic function in $D_R = \{z : |z| < R\}$ such that $|f(z)| < M$ for $z \in D_R$, and $f(0) = 1$. Find a number $0 < \rho < R$, such that $f(z) \neq 0$ for any z with $|z| \leq \rho$.
3. There is an orthognal transformation O of $\mathbb{R}^3$ that transforms the quadratic form

$$q(x, y, z) = 2xy + 2xz + 2yz \tag{2}$$

to the quadratic form

$$Q(X, Y, Z) = \lambda_1 X^2 + \lambda_2 Y^2 + \lambda_3 Z^2, \tag{3}$$

here (X, Y, Z) are the transformed variables. What are the values of $\lambda_1, \lambda_2, \lambda_3$.

4. Let $\{a_1, \ldots, a_n\}$ and c be positive numbers with $c > n$. Use Lagrange multipliers to show that the minimum value of

$$f(x) = \sum_{j=1}^{n} \frac{a_j}{x_j}, \tag{4}$$

on the set

$$S = \left\{ 0 \leq x_j \leq 1 : \sum_{j=1}^{n} \frac{1}{1 - x_j} = c \right\}, \tag{5}$$

is

$$\sum_{j=1}^{n} a_j + \frac{1}{c-n} \left(\sum_{j=1}^{n} a_j^{\frac{1}{2}} \right)^2. \tag{6}$$

5. Suppose that $g(z)$ is an entire function that never vanishes. What are all the possible values of the integrals

$$\int_C \frac{1}{zg(z)} dz, \tag{7}$$

where C is any smooth curve that does not pass through 0 and goes from $z = 1$ to $z = z_0$.

6. In a set of 4000 independent fair coin flips, what is the probability of getting 3000 or more HEADS? Please answer to within a factor of 10. The following common logarithms are accurate to roughly one part in 4000: $\log 2 = 0.301, \log 3 = 0.477$.

7. Let A be a non-singular square matrix ($\det A \neq 0$). Show that there is a polynomial, $p(\lambda) = c_k\lambda^k + \cdots + c_1\lambda + c_0$ such that

$$A^{-1} = c_0 \operatorname{Id} + c_1 A + \cdots + c_k A^k. \tag{8}$$