# AMCS Written Preliminary Exam
# Part I, August 28, 2013

1. Show that the sum
$$f(x) = \sum_{n=1}^{\infty} \sin\left(\frac{x}{n^2}\right)$$
exists for every $x \in [0, \frac{\pi}{2}]$, and that the function is continuous and monotone on this interval.
2. Let $f$ and $g$ be holomorphic functions on the unit disk so that
$$\mathrm{Re}(f(z)) = \mathrm{Re}(g(z))$$
for all $z$ with $|z| < \frac{1}{2}$. Show that $f - g$ is constant and purely imaginary on the unit disk.
3. Identify all the poles, zeros, removable singularities, and essential singularities of the function
$$f(z) = \frac{1}{z} \sin\left(\frac{\pi z}{1 - z}\right)$$
Be sure to consider the behavior as $z \to \infty$.
4. Recall that $f(x) = O(g(x))$ as $x \to \infty$ means there is a constant $C$ such that
$$|f(x)| \leq C|g(x)|$$
for all suffficiently large $x$. Prove that
$$\sum_{k=1}^{n} \log k = n \log n - n + O(\log n).$$
5. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the operator such that $T(x_1, x_2) := (x_2, x_1)$.
   (a) Show that $T$ is self-adjoint with respect to the standard inner product on $\mathbb{R}^2$.
   (b) Consider a new inner product on $\mathbb{R}^2$ such that
   $$\langle x, y\rangle = x_1 y_1 + \frac{1}{2}(x_1 y_2 + x_2 y_1) + \frac{1}{3} x_2 y_2.$$
   Write the matrix of this inner product and compute the adjoint of $T$ with respect to this new inner product (and note that $T^* \neq T$ in this case!).
   (c) Let $\ell : \mathbb{R}^2 \to \mathbb{R}$ be the linear functional $\ell(x_1, x_2) = x_1$ Using the inner product from the part (b), find a vector $v$ such that
   $$\ell(x) = \langle x, v\rangle,$$
   for all $x \in \mathbb{R}^2$.

6. Let $V$ be a complex vector space, with Hermitian inner product $\langle z, w\rangle$. Let $T : V \to V$ be a linear transformation. Show that $T$ is self adjoint if and only if $\langle Tz, z\rangle$ is real for every $z \in V$.
7. What is the probability that a random permutation in $S_4$ (the permutation group on 4 elements) has no fixed points? (A fixed point of a permutation $\sigma$ is a value of $k$ for which $\sigma(k) = k$). For example, for $n = 3$ a permutation with no fixed point is

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}.$$