AMCS Written Preliminary Exam
Part I, August 28, 2012

1. Let

$$f(x) = \begin{cases} (e^{x^2} - e^{-x^2}) \sin\left(\frac{1}{x^3}\right) \text{ for } x \neq 0 \\ 0 \text{ for } x = 0 \end{cases} \tag{1}$$

Show that f is differentiable at 0 and compute $f'(0)$.

2. Give an example of a Riemann integrable function with a countably infinite number of points of discontinuity.

3. For which real values of p is the following series convergent?

$$\sum_{n=2}^{\infty} \frac{1}{(n^2 \log n) \cdot (n^{\frac{1}{n}} - 1)^p}. \tag{2}$$

4. Suppose that A is an $n \times n$ matrix with a 1-dimensional null-space. Show that we can choose vectors u and v so that the linear transformation

$$B = A + u \otimes v^t \tag{3}$$

is invertible. Here $u \otimes v^t x \stackrel{d}{=} \langle x, v \rangle u$. What conditions must u and v satisfy for this to be true?

5. Let A be the $n \times n$ matrix:

$$A = \begin{bmatrix} 0 & 2 & \dots & 2 & 2 \\ 2 & 0 & \dots & 2 & 2 \\ \vdots & \ddots & \ddots & 0 & 2 \\ 2 & & \dots & 2 & 0 \end{bmatrix}$$

(0s along the diagonal and 2s everywhere else.) Compute the eigenvalues and eigenvectors of A.

6. A real valued $\mathscr{C}^2$-function u, defined in the unit disk, is harmonic if it satisfies the partial differential equation $\partial_{xx} u + \partial_{yy} u = 0$. Show that every such function can expressed in the form

$$u(x, y) = f(x + iy) + \overline{f(x + iy)}, \tag{4}$$

where f is an analytic function in the unit disk.

7. A fair coin is flipped until it comes up HEADS. If this occurs on the n^{th} flip, then you win $\$\left(\frac{2}{3}\right)^n$. What is your expected winnings?
8. Let $\{Y_n\}$ be independent random variables uniformly distributed in $[0, 1]$. For $n \geq 1$, let $X_n = Y_1 \cdots Y_n$, the products of the Y_i.
   (a) Compute $E[X_n]$.
   (b) What is the limit, as $n \to \infty$, of $\text{Prob}(X_n > (0.4)^n)$?