AMCS Written Preliminary Exam, I
August 25, 2009

All work should go in the exam booklet, with your final answer clearly marked.

1. Prove that for all $n \in \mathbb{N}$, $n \geq 1$,

$$\frac{1}{2} \leq \sum_{k=1}^{n} \frac{1}{k} - \ln(n) \leq \frac{3}{2}.$$

2. Define a function on the real line by

$$f(x) = \begin{cases} \sqrt{x} \text{ for } x \geq 0 \\ 0 \text{ for } x < 0. \end{cases}$$

   Using the $\epsilon-\delta$-definition of continuity, prove that f is continuous at $x = 0$. Is f uniformly continuous on $\mathbb{R}$? You must prove your answer.

3. Is there a smooth, non-zero 2π-periodic function f, with support of f contained in an interval $[a, b] \subset [0, 2\pi]$, such that $b-a < 2\pi$ and only finitely many Fourier coefficients of f,

$$\hat{f}(n) = \int_{0}^{2\pi} f(x)e^{-inx}dx$$

   are non-zero? Explain your answer.

4. Consider the function $f(x)$ defined on $[0, 1]$ by

$$f(x) = \begin{cases} \frac{1}{q} \text{ if } x = \frac{p}{q}, \text{ with } (p, q) = 1, \\ 1 \text{ if } x \notin \mathbb{Q} \cap [0, 1]. \end{cases}$$

   Is f a Riemann integrable function? You must prove your answer.

5. The Fibonnaci sequence is defined by the conditions $f_0 = 0$, $f_1 = 1$ and $f_{n+2} = f_n + f_{n+1}$. The recurrence relation can be rewritten as

$$\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} f_n \\ f_{n+1} \end{pmatrix} = \begin{pmatrix} f_{n+1} \\ f_{n+2} \end{pmatrix}$$

   Use this relation to show that there is a C so that, for all $n \in \mathbb{N}$, we have:

$$f_n \leq C \left(\frac{1 + \sqrt{5}}{2} \right)^n$$

6. Let $\boldsymbol{a}, \boldsymbol{b}$ be vectors in $\mathbb{R}^3$. The vector cross product defines a linear transformation of $M_{\boldsymbol{a}} : \mathbb{R}^3 \to \mathbb{R}^3$, $M_{\boldsymbol{a}}\boldsymbol{b} = \boldsymbol{a} \times \boldsymbol{b}$, where

$$\boldsymbol{a} \times \boldsymbol{b} \stackrel{d}{=} \det \begin{pmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{pmatrix}.$$

What is the matrix of $M_{\boldsymbol{a}}$ with respect to the standard basis? Prove that for any three vectors, $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$, we have

$$\langle M_{\boldsymbol{a}}\boldsymbol{b}, \boldsymbol{c}\rangle = -\langle \boldsymbol{b}, M_{\boldsymbol{a}}\boldsymbol{c}\rangle.$$

What are the eigenvalues of the matrix $M_{\boldsymbol{a}}$?