AMCS Written Preliminary Exam
Part I, August 26, 2008

1. Prove that the sum
$$\sum_{n=1}^{\infty} \frac{e^{2\pi i n x}}{n}$$
converges for any $x \notin \mathbb{Z}$.
2. Suppose that D is a bounded region in $\mathbb{R}^2$ with a piecewise C^1-boundary. If D is contained in a disk of radius R, then prove the estimate
$$\left| \oint_{\partial D} x\,dy \right| \leq \pi R^2.$$
In this integral ∂D is oriented as the boundary of D. When is this an equality?
3. Prove: If f is a bounded increasing function defined on $(0, 1)$, then f has at most countably many points of discontinuity.
4. Find a conformal map from the semi-circle,
$$\{z : \operatorname{Im} z > 0 \text{ and } |z| < 1\},$$
onto the upper half plane
$$\{z : \operatorname{Im} z > 0\}.$$
5. Let
$$f(z) = \frac{1}{1-z} + \frac{z}{2-z}.$$
Find the Laurent expansions of f valid in
   (a) $1 < |z| < 2$
   (b) $2 < |z|$.
6. Let A be the $n \times n$ matrix:
$$A = \begin{bmatrix} 2 & 1 & \dots & 1 & 1 \\ 1 & 2 & \dots & 1 & 1 \\ \vdots & \ddots & \ddots & 2 & 1 \\ 1 & & \dots & 1 & 2 \end{bmatrix}$$
(2s along the diagonal and 1s everywhere else.) Compute the determinant of A.

7. Let $\mathcal{P}_n$ denote the polynomials of degree at most n. The first derivative defines a linear map from $\mathcal{P}_n$ to itself:

$$D : p \longrightarrow \frac{dp}{dx}.$$

Find the matrix representation of D with respect to the standard basis of monomials $\{x^n, x^{n-1}, \ldots, x, 1\}$. What is the Jordan canonical form of D? Hint: This question can be answered with very little computation.

8. A fair coin is flipped until it comes up HEADS. If this occurs on the n^{th} flip, then you win $\$\left(\frac{3}{2}\right)^n$. What is your expected winnings?