AMCS Written Preliminary Exam
Practice Problems
Adapted from CIMS Written Preliminary Exams
1960-1986

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1 Advanced Calculus

1. Show that, while both limits exist,

$$\lim_{x\to 0}\lim_{y\to 0}\frac{x^2+y^4}{x^2+y^2} \neq \lim_{y\to 0}\lim_{x\to 0}\frac{x^2+y^4}{x^2+y^2}. \tag{1}$$

   State and prove a sufficient condition on a function $f(x, y)$ defined in a neighborhood of $(0, 0)$ so that

$$\lim_{x\to 0}\lim_{y\to 0} f(x, y) = \lim_{y\to 0}\lim_{x\to 0} f(x, y) \tag{2}$$

2. Show that the improper integral

$$I = \int_0^\infty \frac{1}{x}\sin\left(\frac{1}{x}\right) dx \tag{3}$$

   is convergent.

3. Prove that the equation

$$x = e^{-x} \tag{4}$$

   has precisely one root in the interval $[0, 1]$.

4. Compute

$$\lim_{n\to\infty} \frac{1}{n^2} \sum_{k=1}^{n} \sqrt{n^2 - k^2}. \tag{5}$$

5. Prove that

$$\log 2 = 1 - \frac{1}{2} + \frac{1}{3} - \dots \tag{6}$$

6. Let $t(x)$ denote the polynomial of degree 2 consisting of the first three terms of the Taylor series expansion about $x = 1$ of the function $\sqrt{x}$.

   (a) Find $t(x)$.

   (b) Find a positive number δ such that, for all x in the interval, $1 \le x \le 1+\delta$, the inequality

   $$|\sqrt{x} - t(x)| \le 10^{-2} \tag{7}$$

   holds.

7. Consider the equation $ye^y = x$.

   (a) Sketch the of the solution $y = y(x)$.

   (b) Calculate $\frac{dy}{dx}$ as a function of y.

   (c) Give a method for computing the value of y so that $ye^y = 1$.

8. Evaluate the following limits, justifying your answer:

   (a) $\lim_{p\to\infty} \int\limits_0^1 e^{-px}(\cos x)^2 dx$

   (b) $\lim_{t\to\infty} \left(te^t \int\limits_t^\infty \frac{e^{-s}}{s} ds \right)$

9. Suppose that $u(x,t)$ is continuous, together with its first and second partial derivatives; suppose that u and its first partial derivatives are periodic in x of period 1, and suppose that $u_{tt} = u_{xx}$. Prove that

   $$E(t) = \frac{1}{2} \int_0^1 (u_t^2(x,t) + u_x^2(x,t)) dx \tag{8}$$

   is a constant, independent of t.

10. Let f be a continuous function, which maps the closed, bounded interval $[a, b]$ to itself. For $a < x < b$ assume that f is differentiable and $|f'(x)| \leq \theta$, where θ is a fixed number such that $0 < \theta < 1$. Choose $t_1 \in [a, b]$ and define

$$t_{n+1} = f(t_n), \quad n = 1, 2, \dots \tag{9}$$

Prove that the sequence $< t_n >$ has a limit x^*, which satisfies $f(x^*) = x^*$.

11. Compute the following limits

    (a) $\lim_{n\to\infty} \frac{(n+1)^2 n!}{n^n}$

    (b) $\lim_{x\to 0} \frac{x \sin x - x^2}{x^3}$.

12. Prove, justifying all steps that

$$\int_0^\infty e^{-\left(x^2+\frac{a^2}{x^2}\right)} dx = \frac{e^{-2a}\sqrt{\pi}}{2}. \tag{10}$$

You may use without proof that the formula is correct when $a = 0$.

13. Let f be defined in $[0, 1]$ as follows: $f(x) = 0$ if x is irrational. If $x = p/q$, a fraction in lowest terms, then $f(x) = 1/q^2$. Show that f is continuous at x if and only if x is irrational.

14. Determine whether each of the following series is absolutely convergent, conditionally convergent, or divergent:

    (a) $\sum_{n=1}^{\infty} \frac{(-1)^n}{n!}$

    (b) $\sum_{n=1}^{\infty} \frac{n(n+1)}{(n+2)^3}$

    (c) $\sum_{n=1}^{\infty} \left[\left(1+\frac{1}{n}\right)^n - e\right]$

15. (a) What does it mean for a function $f(x)$ to be uniformly continuous on $(-\infty, \infty)$.

    (b) Give an example to show that a function can be uniformly continuous on $(-\infty, \infty)$ without being bounded.

    (c) What can be said about the rate of growth, at infinity, of a uniformly continuous?

16. Prove that

$$\lim_{\lambda\to\infty}\int_1^2 \frac{\cos\lambda t\,dt}{t\sqrt{t-1}} = 0. \tag{11}$$

17. Let f be continuous on the interval $[1,\infty)$ and monotonically decreasing with $\lim_{x\to\infty} f(x) = 0$. Prove that the series

$$\sum_{k=1}^{\infty} f(k) \tag{12}$$

converges if and only if the integral

$$\int_1^{\infty} f(x)dx \tag{13}$$

converges. In the case of convergence which is larger?

18. Prove that

$$\lim_{\epsilon\to 1^-}(1-\epsilon)\sum_{n=0}^{\infty} x_n\epsilon^n = x, \tag{14}$$

if $\lim_{n\to\infty} x_n = x$. Prove that the converse is, in general, false by considering the sequence $x_n = (-1)^n$.

19. Check the following integrals and sums for convergence on divergence

    (a) $\sum_{n=2}^{\infty}\frac{1}{n(\log n)^2}$

    (b) $\sum_{n=1}^{\infty}\left[1-\frac{1}{n}\right]^{n^2}$

    (c) $\int\limits_0^{\infty}\frac{\sqrt{x}}{e^x-1}dx$

    (d) $\int\limits_0^{\infty}\frac{\sin x}{x^2}dx$

20. Introduce new coordinates into the positive quadrant $x > 0,\ y > 0$ be setting

$$\xi = x^2 y \quad \eta = xy^2. \tag{15}$$

    (a) Determine x and y as functions of ξ and η.

(b) Compute the Jacobian determinants

$$A = \det \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \eta} \\ \frac{\partial y}{\partial \xi} & \frac{\partial y}{\partial \eta} \end{bmatrix} \quad B = \det \begin{bmatrix} \frac{\partial \xi}{\partial x} & \frac{\partial \xi}{\partial y} \\ \frac{\partial \eta}{\partial x} & \frac{\partial \eta}{\partial y} \end{bmatrix}. \tag{16}$$

(c) Check that $AB = 1$. Is this a general fact for such change of variable, or merely an accident in this case? Prove or disprove.

21. Compute the outward-pointing normal vector $\boldsymbol{n}$ at a point (x, y) on the ellipse:

$$E = \{(x, y) : \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\}. \tag{17}$$

Verify, by direct computation, the correctness of the divergence theorem:

$$\iint_{\text{inside } E} \text{div} X dA = \int_E X \cdot \boldsymbol{n} dl, \tag{18}$$

where $X = (y, x)$.

22. Show that there exist constants c_1, c_2 such that, for all integers $N \geq 1$ we have that

$$\left| \sum_{n=1}^{N} \frac{1}{\sqrt{n}} - 2\sqrt{N} - c_1 \right| \leq \frac{c_2}{\sqrt{N}}. \tag{19}$$

23. For

$$f(x) = \sum_{n=1}^{\infty} \frac{nx^n}{1 - x^n}, \tag{20}$$

prove that:

(a) The series converges for all x in the interval $(-1, 1)$.

(b) For $x \in [0, 1)$ we have that $(1 - x)^2 f(x) \geq x$.

24. Let $f(x)$ be a real valued function with continuous derivatives on $[0, 1]$; such that $|f(1)| \geq |f(0)|$. Show that either there is an $x \in (0, 1)$, such that $f(x)$ and $f'(x)$ have the same sign, or $f(x) \equiv$ constant.

25. Find the shortest distance from the ellipse

$$\frac{x^2}{4} + y^2 = 1, \tag{21}$$

to the straight line $x + y = 4$.

26. Show that for $f(x)$ a continuous function on [0, 1] we have

$$\lim_{n\to\infty} \sqrt{n} \int_0^1 \frac{f(x)\,dx}{1+nx^2} = \frac{\pi}{2} f(0). \tag{22}$$

2 Linear Algebra

1. Consider the system of linear equations

$$\begin{aligned} kx + y + z &= 1 \\ x + ky + z &= 1 \\ x + y + kz &= 1. \end{aligned} \tag{23}$$

   Determine for which numbers k the system has (a) no solution, (b) a unique solution, (c) more than one solution.

2. For each of the following conditions on a real square matrix a_{ij} find a matching condition on its eigenvalues:

________ (a) $a_{ij} = a_{ji}$	(A) all non-zero eigenvalues are pure imaginary
________ (b) $\det(a_{ij}) = 0$	(B) all eigenvalues are zero
________ (c) $a_{ij} = 0$ if $i \geq j$	(C) all eigenvalues are real
________ (d) $\sum_j a_{ij}a_{jk} = \delta_{ik}$	(D) no eigenvalue is zero
________ (e) $\det(a_{ij}) \neq 0$	(E) at least one eigenvalue is zero
________ (f) $a_{ij} = -a_{ji}$	(F) each real negative eigenvalue is -1

3. Introduce a scalar product on the space of continuous, real valued functions defined on [0, 1] by setting

$$\langle f, g\rangle = \int_0^1 x f(x) g(x) dx. \tag{24}$$

   Find an orthonormal basis for the subspace spanned by 1, x, and x^2.

4. Find an orthogonal transformation that reduces the quadratic form

$$q(x, y, z) = x^2 - y^2 + z^2 + 2xy$$

to the form

$$Q(X, Y, Z) = aX^2 + bY^2 + cZ^2.$$

What are the values of a, b, c?

5. Suppose that A is an $n \times n$ matrix of complex numbers. Prove that is λ is an eigenvalue of A^2, then $\sqrt{\lambda}$ or $-\sqrt{\lambda}$ is an eigenvalue of A.

6. Let U be the subspace of $\mathbb{R}^4$ spanned by the vectors

$$\begin{aligned} a_1 =& (1, 1, 0, 0) \\ a_2 =& (0, 1, 1, 0) \\ a_3 =& (0, 0, 1, 1) \end{aligned} \tag{25}$$

and let V be the subspace spanned by

$$\begin{aligned} b_1 =& (1, -1, 0, 0) \\ b_2 =& (0, 1, -1, 0) \\ b_3 =& (0, 0, 1, -1) \end{aligned} \tag{26}$$

Determine the dimension of the intersection $W = U \cap V$, and find bases of U and V, parts of which are basis vectors for W.

7. Find the inverse of the matrix

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{27}$$

8. Let A be a 3×3 matrix such that $A^2 = A$. Find all possible Jordan canonical forms for A.

9. Suppose that a square $n \times n$ matrix A commutes with all diagonal matrices. What can one say about the matrix A.

10. What is the dimension (over the real numbers) of all <u>odd</u> polynomials (i.e., $p(-x) = -p(x)$) of degree at most 7. Prove that if a_0, a_1, b_0, b_1 are arbitrary real numbers, then there exists a unique such polynomial $p(x)$ satisfying

$$\begin{aligned} p(1) = a_0 \quad & \frac{dp}{dx}(1) = a_1 \\ p(2) = b_0 \quad & \frac{dp}{dx}(2) = b_1. \end{aligned} \tag{28}$$

11. Determine the eigenvalues and eigenvectors of the matrix

$$A = \begin{pmatrix} 1 & 0 & -2 \\ 0 & 0 & 0 \\ -2 & 0 & 4 \end{pmatrix}. \tag{29}$$

12. Let A be a real symmetric matrix and form the matrix

$$R(z) = (z\,\mathrm{Id} - A)^{-1} \tag{30}$$

for complex values of z, whenever it is defined. Prove: The elements of $R(z)$ are quotients of two polynomials in z whose denominators have zeros of at most first order.

13. Consider the vector space of functions of the form

$$p(t) = a + b\cos t + c\sin t \text{ where } a, b, c \in \mathbb{R}, \tag{31}$$

with the norm

$$N(p) = \sqrt{a^2 + \frac{1}{2}(b^2 + c^2)}. \tag{32}$$

Define the linear transformation $Tp(t) = p(t + \alpha)$, where α is a given real number. Show that the transformation T is orthogonal and compute its eigenvalues.

14. Suppose that A is a non-singular linear transformation of an n-dimensional linear space into itself. Show that there exists some polynomial $c_0 + c_1 z + \cdots + c_k z^k$ so that

$$A^{-1} = c_0\,\mathrm{Id} + c_1 A + \cdots + c_k A^k. \tag{33}$$

15. In $\mathbb{R}^3$ suppose that the plane Q goes through the points

$$(0, 0, 0) \quad (1, 2, 3) \quad (-1, 0, 1). \tag{34}$$

Let P be the plane parallel to Q which goes through the point $(3, 7, -9)$. Find the distance of the plane P to the origin.

16. (a) What does it mean to say that n vectors $r_1, \ldots, r_n$ are linearly independent?

    (b) Are the vectors (1, 1, 1), (1, 0, 1), (0, 1, 0) linearly independent?

    (c) Are the vectors (1, 2, 3), (3, 1, 2), (2, 3, 1) linearly independent?

17. Suppose that V is a vector space over the real numbers. What does it mean to say that "the dimension of V is n"? If V is also a vector space over the complex numbers, how is the dimension of V as a real vector space related to its dimension as a complex vector space. Give an example of a vector space that is both a real and a complex vector space.

18. Suppose A is a positive definite, real symmetric matrix. That is, the quadratic form

$$Q(x) = \langle Ax, x \rangle \tag{35}$$

is positive definite. Prove the Schwarz inequality:

$$\langle Ax, y \rangle^2 \leq \langle Ax, x \rangle \langle Ay, y \rangle. \tag{36}$$

19. Let A and B be two $n \times n$ matrices. Prove that the eigenvalues of AB are the same as the eigenvalues of BA.

20. Sum the infinite series

$$A = \sum_{n=0}^{\infty} \frac{1}{n!} \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}^n . \tag{37}$$

Note that

$$\begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} = 2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} . \tag{38}$$

21. Find the minimal polynomial of the matrix (a_{ij}) when the matrix elements a_{ij} have the form $a_{ij} = u_i v_j$. Note: You must state precisely any standard theorem you use.

22. Suppose that A is a Hermitian symmetric $n \times n$ matrix of complex numbers all of whose eigenvalues lie inside the interval $(-1, 1)$. Prove that the matrix $A^3 + \mathrm{Id}$ is positive definite.

23. If all the main diagonal entries of a symmetric matrix are all positive, can all the eigenvalues be negative? (Give a reason for you answer).

24. Determine the eigenvalues and corresponding eigenvectors of the matrix A

$$A = \begin{pmatrix} 4 & 1 \\ 9 & -4 \end{pmatrix} . \tag{39}$$

Compute

$$A^{99} \begin{pmatrix} 5 \\ -15 \end{pmatrix} . \tag{40}$$

25. Determine the values of x such that if

$$A = \begin{pmatrix} x & x-1 \\ x+1 & x \end{pmatrix}, \tag{41}$$

then there exists a diagonal matrix D and a non-singular matrix B with

$$B^{-1}AB = D. \tag{42}$$

In such cases find B and D.

3 Complex Analysis

1. Suppose that $\sum_{n=0}^{\infty} a_n(z-1)^n$ is the power series of the function $\frac{1}{\cos z}$ about the point $z = 1$. Does the series $\sum_{n=0}^{\infty} |a_n|$ converge or diverge? Justify your answer.

2. (a) Is $e^x(\cos y + i \sin y)$ an analytic function of $z = x + iy$?

   (b) Is $e^x(\sin y + i \cos y)$ an analytic function of $z = x + iy$?

   (c) Is $e^x \sin y$ the real part of an analytic function of $z = x + iy$?

   Justify your answers.

3. Using the calculus of residues, evaluate the integral

$$\int_{-\infty}^{\infty} \frac{dx}{(x^2+1)^3}. \tag{43}$$

4. Map the semi-circle $|z| < 1$, $\operatorname{Im} z > 0$, conformally onto the first quadrant, $\operatorname{Re} w > 0$, $\operatorname{Im} w > 0$. Is the mapping $w = w(z)$ unique?

5. What is the dimension of the space of homogeneous polynomials in x, y, of degree n, which satisfy

$$\frac{\partial^2 P_n}{\partial x^2} + \frac{\partial^2 P_n}{\partial y^2} = 0? \tag{44}$$

6. Find a region in the complex plane where the function e^{z^2} assumes every non-zero, complex value exactly once.

7. Find three power or Laurent series expansions for

$$f(z) = \frac{1}{z^2 - 1} \tag{45}$$

such that every point except $z = \pm 1$ is a point of absolute convergence for at least one of these series.

8. Evaluate the integrals

$$\text{(a)} \int_{-\infty}^{\infty} \frac{e^{ix} dx}{1 + x^2}, \qquad \text{(b)} \lim_{a \to \infty} \int_{-a}^{a} \frac{dx}{1 + x}. \tag{46}$$

9. Let $f(z)$ be analytic in $|z| < 1.1$, and assume that $|f(z)| \leq 1$ on $|z| = 1$. Prove that

$$f(z) + 8z^2 - 2 = 0 \tag{47}$$

has two roots in the unit circle.

10. Suppose that $f(z)$ is an entire function and $|f(z)| \leq e^x$ $(z = x + iy)$ throughout the complex plane. What can be said about $f(z)$?

11. Find a holomorphic function $w = f(z)$ mapping the sector $|\arg z| < \alpha < \pi$ conformally onto the unit circle $|w| < 1$. Describe the behavior of $f(z)$ near $z = 0$.

12. Let $\sum_{n=0}^{\infty} a_n z^n$, $\sum_{n=0}^{\infty} b_n z^n$ have radii of convergence r_a, r_b. What can be said about the radii of convergence of the series

$$\text{(a)} \sum_{n=0}^{\infty} (a_n + b_n) z^n, \qquad \text{(b)} \sum_{n=0}^{\infty} a_n b_n z^n? \tag{48}$$

13. Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ have a positive radius of convergence. Does there exist a series $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$, satisfying

$$f(g(z)) = z? \tag{49}$$

Does this series have a positive radius of convergence? Are the coefficients $\{b_n\}$ uniquely determined?

14. How many roots does the equation

$$\frac{1}{2}e^z + z^4 + 1 = 0 \tag{50}$$

have in the left half plane $\operatorname{Re} z < 0$?

15. For each condition below give an example of a function analytic in $0 < |z| < 1$ that:

    (a) Has a simple pole at $z = 0$ and vanishes at $z = \frac{1}{2}$.

    (b) Has an essential singularity at $z = 0$ and a pole of order at $z = 1$.

    (c) Has a removable singularity at $z = 0$ and an essential singularity at $z = i$.

16. Suppose that F is analytic in $|z| < 10$ and $\operatorname{Im} F(e^{i\theta}) = \sin\theta$, for $\theta \in \mathbb{R}$. Find F in $|z| < 10$ and justify your answer.

17. Where does the series

$$\sum_{n=1}^{\infty} \frac{z^n}{n(n+1)} \tag{51}$$

converge? Why? Express this sum in terms of elementary functions.

18. Represent all complex values of $(-1)^i$ and $(1+i)^{\frac{2}{3}}$ in the form $a + bi$.

19. State the argument principle. Use it to prove the fundamental theorem of algebra: A polynomial of degree exactly $n > 0$ has n complex roots, counted with multiplicity.

20. Find a function that conformally maps the interior of the unit disk onto the strip $-\pi < \operatorname{Im} w < \pi$, so that $z = 0$ goes into $w = 1$.

21. Find the radii of convergence of the the following series, justifying your answers

    (a) $\sum_{n=1}^{\infty} (n^{-1}z)^n$

    (b) $\sum_{n=1}^{\infty} (nz)^n$

    (c) $\sum_{n=1}^{\infty} \left(\frac{z \log n}{n}\right)^n$

22. Show that if $|f(z)| < M$ on $|z| = 1$ and $f(z) = z^2 g(z)$, then

$$|f(z)| \leq \frac{M}{4} \text{ for } |z| \leq \frac{1}{2}. \tag{52}$$

Here $f(z)$ and $g(z)$ are analytic in $|z| \leq 1$.

23. (a) What is the angle between the curves $\mathrm{Re}(z^3) = 1$ and $\mathrm{Re}(z^3) = \mathrm{Im}(z^3)$?

    (b) What is the angle between the curves $\mathrm{Re}(z^3) = 0$ and $\mathrm{Re}(z^3) = \mathrm{Im}(z^3)$?

24. Suppose that $f(z)$ is an entire function in the plane satisfying

$$|f(z)| \leq M(1+|z|)^n. \tag{53}$$

Prove that f is a polynomial of degree at most n.

25. Consider the power series expansion

$$f(z) = \frac{\sinh z}{2\cosh z - 1} = \sum_{n=0}^{\infty} \beta_n z^n, \tag{54}$$

in a neighborhood of $z = 0$. Find $\limsup_{n\to\infty} |\beta_n|^{\frac{1}{n}}$. Compute the first three terms of the Laurent expansion of $1/f(z)$ about $z = 0$.

4 Discrete Math (adapted from D'Angelo and West, *Mathematical Thinking*)

1. Let $< a_n >$ be a sequence satisfying $a_n = 2a_{n-1} + 3a_{n-2}$ for $n \geq 3$.

    (a) Given that a_1, a_2 are odd integers, prove that a_n is odd for all $n \in \mathbb{N}$.

    (b) Given that $a_1 = a_2 = 1$, prove that $a_n = \frac{1}{2}(3^{n-1} - (-1)^n)$, for all $n \in \mathbb{N}$.

2. Let $N = \{1, 2, 3, \ldots, n\}$. Show that N has as many subsets with an even number of elements as subsets with an odd number of elements.

3. A fair coin is flipped exactly $2n$ times. Compute the probability of obtaining exactly n heads. Evaluate the formual for $n = 10$.

4. We roll a six sided die three times. Determine the probability that the sum of the values rolled equals eleven.

5. Compute the probability that a random five-card hand has the following:

    (a) At least three cards with the same rank.

    (b) At least two cards with the same rank.

6. Let X_1, X_2, X_3 be independent random variables, each taking values in the set $\{1, 2, \ldots, n\}$, with each value equally probable. Compute the probability that $X_1 + X_2 + X_3 \leq 6$, given that $X_1 + X_2 \geq 4$.

7. Suppose that Mary and Jim and n other people stand in a line in random order. For each k with $0 \leq k \leq n$ find the probability that exactly k people stand between Mary and Jim.

8. Suppose that X is a random variable that takes values in $\{1, \ldots, n\}$. Prove that

$$E(X) = \sum_{k=1}^{n} P(X \geq k). \tag{55}$$

9. The Vandermonde convolution theorem states that for any three nonnegative integers n, m, k,

$$\sum_{j=0}^{k} \binom{n}{j} \binom{m}{k-j} x = \binom{n+m}{k}. \tag{56}$$

Prove (56) by either

(A) using the following generating function identity for the binomial coefficients

$$\sum_{j=0}^{k} \binom{n}{j} = (1+x)^n, \tag{57}$$

or

(B) by combinatorial reasoning, using the fact that $\binom{n}{j}$ equals the number of j-element subsets of a set of n different objects.

10. A wheel of fortune has the integers from 1 to 25 placed on it in a random manner. Show that regardless of how the numbers are positioned on the wheel, there are three adjacent numbers whose sum is at least 39.

11. Use mathematical induction to prove that the following identity holds for any positive integer n.

$$\sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}. \tag{58}$$

12. Determine the number of positive integers n, $1 \leq N \leq 2000$, that are

a) not divisible by 2, 3, or 5.

b) not divisible by 2, 3, or 5, but are divisible by 7.

13. Determine how many integer solutions there are to

$$x_1 + x_2 + x_3 + x_4 = 19, \tag{59}$$

if $0 \leq x_i < 12$ for $1 \leq i \leq 4$.

14. A *palindrome* of a positive integer n is a composition of n (a representation of n as a sum of positive integers), which reads the same backwards as forwards. For example, there are 4 palindromes of 5, namely

$$\begin{aligned} 5 \\ 1+3+1 \\ 2+1+2 \\ 1+1+1+1+1. \end{aligned}$$

If p_n denotes the number of palindromes of n, prove that $p_n = 2p_{n-2}$.

15. Suppose that X is a random variable with mean $E(X) = \bar{X}$ and variance $\sigma = E(|X - \bar{X}|^2)$. Prove Chebyshev's inequality:

$$\text{Prob}\{|X - \bar{X}| \geq t\} \leq \frac{\sigma^2}{t^2}. \tag{60}$$

16. Let X denote the outcome of flipping a fair coin, with $X = 1$ if the coin comes up heads and $X = 0$ for tails. If $X_1, X_2, \ldots$ are the outcomes of independent tosses of the coin, show that

$$\text{Prob}\{X_1 + \cdots + X_N = j\} = \binom{N}{j}\frac{1}{2^N}. \tag{61}$$

What are $E(X_1 + \cdots + X_N)$ and $\text{Var}(X_1 + \cdots + X_N)$?