Version 2: AMCS Written Preliminary Exam  
Part II, Aug, 2021

7. Suppose \( x_0 = \frac{1}{2} = y_0 \) and for each \( n \geq 1 \),
\[
\begin{align*}
x_n &:= x_n^2 \cos x_n - y_n^2 \sin x_n \\
y_n &:= x_n^2 \sin x_n + y_n^2 \cos x_n
\end{align*}
\]
Prove that \((x_n, y_n)\) converges in \( \mathbb{R}^2 \) to some limit point \((x, y)\) as \( n \to \infty \).

8. Let \( g \) be a bounded real-valued function on \([-1, 1]\) which is continuous at all points other than \( x = 0 \) and satisfies \( g(0) = 1 \), \( \lim_{x \to 0^+} g(x) = 2 \), and \( \lim_{x \to 0^-} g(x) = 3 \). Compute the limit
\[
\lim_{n \to \infty} n \int_{-1}^{1} x(1 - x^2)^{n-1} g(x) dx
\]
and prove that your answer is correct.

9. Suppose \( f_1, f_2, f_3 : \mathbb{R} \to \mathbb{R} \) are each twice differentiable and consider the matrix-valued function
\[
W = \begin{pmatrix}
f_1 & f_2 & f_3 \\
f_1' & f_2' & f_3' \\
f_1'' & f_2'' & f_3''
\end{pmatrix}.
\]
Show that if there is some \( x \in \mathbb{R} \) at which the determinant of \( W \) is nonzero, then \( f_1, f_2, f_3 \) are linearly independent over the real numbers (i.e., for any real numbers \( c_1, c_2, c_3 \) such that \( c_1 f_1(x) + c_2 f_2(x) + c_3 f_3(x) \equiv 0 \) for all \( x \in \mathbb{R} \) it must be the case that \( c_1 = c_2 = c_3 = 0 \)).
10. Suppose that $A = (a_{ij})$ is an $n \times n$ complex matrix such that
\[ \sum_{i \neq j} |a_{ij}| < |a_{jj}| \text{ for each } j = 1, \ldots, n. \]

Can $A$ be singular? If yes, give an example, and if no, give a proof.

11. Give a complete description of the collection of all functions $g : \mathbb{C} \setminus \{0\} \to \mathbb{C}$ which are analytic and satisfy $|g(z)| > |z|^{-\frac{2}{\gamma}}$ at every point of $\mathbb{C} \setminus \{0\}$.

12. A collection of balls is sorted into bins as follows: Ball #1 goes in bin 1. Ball #2 is placed either in bin 1 or bin 2 with equal probability. As each new ball is sorted, the rule followed is that, if at that moment the occupied bins are $\{1, \ldots, k\}$, then the ball has probability $\frac{1}{k+1}$ of being placed in any particular one of the occupied bins and also has probability $\frac{1}{N_k}$ of being placed in the unoccupied bin number $k + 1$. For each $k > 1$, let $N_k$ be the total number of balls in bins 1, $\ldots$, $k - 1$ at the moment when bin $k$ receives its first ball. What is $\mathbb{E}[N_{10}]$? Hint: What is $\mathbb{E}[N_k - N_{k-1}]$?