Version 2: AMCS Written Preliminary Exam Part II, Aug, 2021

7. Suppose $x_0 = \frac{1}{2} = y_0$ and for each $n \geq 1$,

$$x_n := x_n^2 \cos x_n - y_n^2 \sin x_n$$
$$y_n := x_n^2 \sin x_n + y_n^2 \cos x_n$$

Prove that (x_n, y_n) converges in $\mathbb{R}^2$ to some limit point (x, y) as $n \to \infty$.

8. Let g be a bounded real-valued function on $[-1, 1]$ which is continuous at all points other than $x = 0$ and satisfies $g(0) = 1$, $\lim_{x \to 0^+} g(x) = 2$, and $\lim_{x \to 0^-} g(x) = 3$. Compute the limit

$$\lim_{n \to \infty} n \int_{-1}^{1} x(1 - x^2)^{n-1} g(x) dx$$

and prove that your answer is correct.

9. Suppose $f_1, f_2, f_3 : \mathbb{R} \to \mathbb{R}$ are each twice differentiable and consider the matrix-valued function

$$W = \begin{pmatrix} f_1 & f_2 & f_3 \\ f_1' & f_2' & f_3' \\ f_1'' & f_2'' & f_3'' \end{pmatrix}.$$

Show that if there is some $x \in \mathbb{R}$ at which the determinant of W is nonzero, then f_1, f_2, f_3 are linearly independent over the real numbers (i.e., for any real numbers c_1, c_2, c_3 such that $c_1 f_1(x) + c_2 f_2(x) + c_3 f_3(x) \equiv 0$ for all $x \in \mathbb{R}$ it must be the case that $c_1 = c_2 = c_3 = 0$).

10. Suppose that $A = (a_{ij})$ is an $n \times n$ complex matrix such that

$$\sum_{i \neq j} |a_{ij}| < |a_{jj}| \text{ for each } j = 1, \ldots, n.$$

Can A be singular? If yes, give an example, and if no, give a proof.

11. Give a complete description of the collection of all functions $g : \mathbb{C} \setminus \{0\} \to \mathbb{C}$ which are analytic and satisfy $|g(z)| > |z|^{-\frac{7}{3}}$ at every point of $\mathbb{C} \setminus \{0\}$.

12. A collection of balls is sorted into bins as follows: Ball #1 goes in bin 1. Ball #2 is placed either in bin 1 or bin 2 with equal probability. As each new ball is sorted, the rule followed is that, if at that moment the occupied bins are $\{1, \ldots, k\}$, then the ball has probability $\frac{1}{k+1}$ of being placed in any particular one of the occupied bins and also has probability $\frac{1}{k+1}$ of being placed in the unoccupied bin number $k + 1$. For each $k > 1$, let N_k be the total number of balls in bins $1, \ldots, k-1$ at the moment when bin k receives its first ball. What is $\mathbb{E}[N_{10}]$? Hint: What is $\mathbb{E}[N_k - N_{k-1}]$?