Version 2: AMCS Written Preliminary Exam Part I, Aug, 2021

1. Let $g : \mathbb{R} \to \mathbb{R}$ be twice continuously differentiable and satisfy

$$\int_{-\infty}^{\infty} |g''(x)| dx < \infty.$$

Prove that g is uniformly continuous on $\mathbb{R}$.

2. Suppose $\{f_n\}_{n=1}^{\infty}$ is a sequence of real-valued functions on $[0, 1]$ such that each f_n is monotone increasing on the interval $[0, 1]$. If $\lim_{n\to\infty} f_n(0) = \lim_{n\to\infty} f(1) = A$, show that f_n converges uniformly to A on the entire interval $[0, 1]$.

3. Compute a basis of $\mathbb{R}^3$ in which the matrix

$$\begin{bmatrix} 1 & 3 & -2 \\ 1 & -1 & 2 \\ 1 & 3 & -2 \end{bmatrix}$$

is diagonal. Your change of basis matrix will not be an orthogonal matrix; explain why no orthonormal basis can diagonalize this matrix.

4. Suppose N is a complex $n \times n$ matrix. Show that $N^k = 0$ for some $k \geq 1$ if and only if there is a basis of $\mathbb{C}^n$ such that, when reexpressed in this basis, N is upper-triangular with diagonal entries all equal to zero.

5. Compute the integral

$$\int_{-\infty}^{\infty} \frac{\cos x - 1}{x^2} dx.$$

6. Suppose $\tau_1, \tau_2, \tau_3, \dots$ is a sequence of nonnegative, independent random variables each with the same probability density function $f(t) = e^{-t}$ for $t \geq 0$. Use induction to verify that for each n, the probability density function for

$$S_n := \tau_1 + \cdots + \tau_n$$

is

$$g_n(t) = \frac{t^{n-1}}{(n-1)!} e^{-t}.$$