7. Let

\[ \Phi(x, y) := \left( \frac{x^3}{5} + \frac{\sin y}{5}, \frac{\cos x}{5} + \frac{y^3}{5} \right). \]

Show that there is a unique point \((x, y) \in [0, 1]^2\) such that \(\Phi(x, y) = (x, y)\).

8. Let \(f(x)\) be a continuous function on \(\mathbb{R}\) and let

\[ f_n(x) := \int_{-\frac{1}{n}}^{\frac{1}{n}} (n - n^2|y|) f(x - y)dy \]

where \(n\) is any positive integer. Prove that, as \(n \to \infty\), \(f_n(x)\) converges to \(f(x)\) uniformly on any bounded interval of \(\mathbb{R}\).

9. Prove that the following is an inner product on the real vector space \(V\) polynomials of degree at most 2 and compute an example of a basis of \(V\) which is orthonormal with respect to this inner product.

\[ \langle p, q \rangle := p(0)q(0) + \int_{-1}^{1} p(x)q(x)dx. \]
10. Suppose that $A = (a_{ij})$ is an $n \times n$ complex matrix such that

$$\sum_{i=1}^{n} \sum_{j=1}^{n} |a_{ij}|^2 \leq 1.$$ 

Show that all eigenvalues of $A$ lie in the closed unit disk $\{z \in \mathbb{C} : |z| \leq 1\}$ and that for any $r < 1$, there is an $A$ of the sort described above which has at least one eigenvalue $\lambda$ satisfying $|\lambda| \geq r$.

11. Find all zeros, poles, and essential singularities of the function

$$\frac{e^{\pi z} + 1}{e^{\pi/z} + 1}$$

in the complex plane. Compute the order of each zero or pole, and for each pole, compute the residue as well.

12. Let $C \subset \mathbb{R}^2$ be the circle $x^2 + y^2 = 4$. If $u$ and $v$ are two points chosen independently and uniformly at random on $C$, what is the probability that the line segment joining $u$ and $v$ will be entirely contained in the annulus $A$ given by $3 \leq x^2 + y^2 \leq 4$?