AMCS Written Preliminary Exam
Part II, May 3, 2021

7. Let

$$\Phi(x,y) := \left(\frac{x^3}{5} + \frac{\sin y}{5}, \frac{\cos x}{5} + \frac{y^3}{5}\right).$$

Show that there is a unique point $(x,y) \in [0,1]^2$ such that $\Phi(x,y) = (x,y)$.

8. Let $f(x)$ be a continuous function on $\mathbb{R}$ and let

$$f_n(x) := \int_{-\frac{1}{n}}^{\frac{1}{n}} (n - n^2|y|) f(x-y)dy$$

where n is any positive integer. Prove that, as $n \to \infty$, $f_n(x)$ converges to $f(x)$ uniformly on any bounded interval of $\mathbb{R}$.

9. Prove that the following is an inner product on the real vector space V polynomials of degree at most 2 and compute an example of a basis of V which is orthonormal with respect to this inner product.

$$\langle p, q \rangle := p(0)q(0) + \int_{-1}^{1} p(x)q(x)dx.$$

10. Suppose that $A = (a_{ij})$ is an $n \times n$ complex matrix such that

$$\sum_{i=1}^{n} \sum_{j=1}^{n} |a_{ij}|^2 \leq 1.$$

Show that all eigenvalues of A lie in the closed unit disk $\{z \in \mathbb{C} \;:\; |z| \leq 1\}$ and that for any $r < 1$, there is an A of the sort described above which has at least one eigenvalue λ satisfying $|\lambda| \geq r$.

11. Find all zeros, poles, and essential singularities of the function

$$\frac{e^{\pi z} + 1}{e^{\pi / z} + 1}$$

in the complex plane. Compute the order of each zero or pole, and for each pole, compute the residue as well.

12. Let $C \subset \mathbb{R}^2$ be the circle $x^2 + y^2 = 4$. If u and v are two points chosen independently and uniformly at random on C, what is the probability that the line segment joining u and v will be entirely contained in the annulus A given by $3 \leq x^2 + y^2 \leq 4$?