AMCS Written Preliminary Exam
Part I, May 3, 2021

1. Let $f : [0,1] \to \mathbb{R}$ be continuously differentiable with $f(0) = 0$. Prove that

$$\sup_{0\le x\le 1} |f(x)| \le \sqrt{\int_0^1 |f'(x)|^2 dx}.$$

2. Suppose $\{f_n\}_{n=1}^{\infty}$ is a sequence of real-valued functions on $\mathbb{R}$ which converges uniformly to some limit function f. If $\lim_{n\to\infty} \lim_{x\to\infty} f_n(x)$ exists, prove that

$$\lim_{x\to\infty} \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \lim_{x\to\infty} f_n(x)$$

(and in particular, show that the limit on the left-hand side exists).

3. Let

$$M := \begin{bmatrix} \frac{3}{2} & \frac{3}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{3}{2} & \frac{3}{2} \end{bmatrix}.$$

Find a 2×2 matrix A and a 2×4 matrix B, both of which have orthonormal rows, and a diagonal 2×2 matrix D with positive diagonal entries such that $M = ADB$.

4. Suppose A and B are $n \times n$ complex matrices which commute.

   a. Show that
   $$(A+B)^j = \sum_{k=0}^{j} \binom{j}{k} A^k B^{j-k}$$
   for every integer $j \geq 1$.

   b. Prove that
   $$e^{A+B} = e^A e^B.$$

5. Evaluate
$$\int_{-\infty}^{\infty} \frac{dx}{1+x^{2n}}$$
where n is a positive integer. Hint: The simplest approach is to evaluate a contour integral around a wedge-shaped region containing only a single pole.

6. For each integer $n \geq 1$, suppose $x_n \in (0, \infty)$ is randomly selected with pdf $\rho_k(x)$ given by
$$\rho_k(x) := \frac{k}{x^{k+1}} e^{-1/x^k}.$$
Assuming that the x_n are chosen independently,

   a. What is
   $$\mathbb{P}\left(\sup_{n\geq 1} x_n \leq \frac{1}{2}\right)?$$

   b. What is
   $$\mathbb{P}\left(\sup_{n\geq 1} x_n \leq 2\right)?$$