AMCS Written Preliminary Exam
Part II, August 28, 2020

1. Let x_1, x_2, x_3, and x_4 be real numbers satisfying $x_1 + x_2 + x_3 + x_4 = \pi$. Show that
$$\sin x_1 \sin x_2 \sin x_3 \sin x_4 \leq \frac{1}{4}.$$

2. Let $f(x), x \in \mathbb{R}$ be a function that is bounded and whose first and second derivative are also bounded. For any bounded continuous function $u(x), x \in \mathbb{R}$, let:
$$||u|| := \sup_{x\in\mathbb{R}} |u(x)|.$$
   (a) For any x and h, prove that:
$$|f(x+h) - f(x) - f'(x)h| \leq \frac{1}{2}||f''||h^2.$$
   (b) Prove that
$$|f'(x)| \leq \frac{2}{h}||f|| + \frac{h}{2}||f''||.$$
   From this, show that
$$||f'|| \leq 2\sqrt{||f||||f''||}.$$

3. Compute the Laurent series expansion for
$$f(z) := \frac{1}{2z^2 + 3z - 2}$$
valid in an annular region containing $z = 1$.

4. Suppose f is an entire function which satisfies
$$\int_0^{2\pi} |f(re^{i\theta})| d\theta \leq r^{17/3}.$$
for all $r > 0$. Prove that f is the zero function.

5. Suppose A and B are $n \times n$ matrices and $AB = BA$, and $(A - B)^k = 0$ for some $k > 0$. Prove that, if λ is an eigenvalue of A, then it is also an eigenvalue of B.

6. Prove or disprove: given any four complex numbers z_1, z_2, z_3, z_4, there is a complex polynomial f of degree at most three such that $f(1) = z_1, f(2) = z_2, f(3) = z_3$ and $f(4) = z_4$.

7. The integers from 1 to N are rearranged uniformly at random. A position k is called a left-to-right maximum if the number in that position is greater than the $k - 1$ numbers to its left. What is the expected number of $k < N$ such that both k and $k + 1$ are left-to-right maxima?