AMCS Written Preliminary Exam
Part I, August 28, 2020

1. Find the set of points $E \subset \mathbb{R}$ at which the following series converges:

$$\sum_{n=1}^{\infty} \frac{nx^n}{n^2 + x^{2n}},$$

then find an open interval $I \subset E$ on which the series converges to a continuous function and justify your answer.

2. Use residue calculus to compute the integral

$$\int_{-\infty}^{\infty} \frac{dx}{x^4 + 3x^2 + 4}.$$

3. Let $f : \mathbb{R} \to \mathbb{R}$ be uniformly continuous with $f(0) = 0$. Prove that there exists a positive number B such that $|f(x)| \leq 1 + B|x|$ for all $x \in \mathbb{R}$.

4. Let f be a rational function in the complex plane which has no poles in the half plane $\mathrm{Im}(z) \geq 0$. Prove that

$$\sup_{\mathrm{Im}(z) \geq 0} |f(z)| = \sup_{\mathrm{Im}(z)=0} |f(z)|.$$

5. Consider the matrices

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \text{and} \qquad B = \begin{bmatrix} -1 & 0 & 0 & 0 \\ -1 & 1 & 1 & -1 \\ -1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \end{bmatrix}.$$

Does there exist an invertible real matrix P such that $B = P^{-1}AP$? Justify your answer.

6. Suppose v is a column vector in $\mathbb{R}^2$ and M is a 2×2 matrix with characteristic polynomial $t^2 - 6t + 9$. What are the possible asymptotic behaviors of $||M^n v||$ as $n \to \infty$?

7. Two independent draws are added together from a probability distribution on the positive reals having a continuous density with a value of 3 at zero. What is the density at zero of the resulting random variable?