AMCS Written Preliminary Exam
Part I, August 27, 2014

1. Let x_0 be a real number; we define a sequence of real numbers by setting:
$$x_{n+1} = x_n - \frac{x_n^3 + x_n^2 - 2x_n - 2}{3x_n^2 + 2x_n - 2}.$$
Suppose that $x_0 > 1$. Show that the sequence has a limit, independent of $x_0 > 1$, and determine what it is.
2. Prove that $\sin x \leq x$ for all $0 \leq x$.
3. What is the radius of convergence of the power series:
$$f(z) = \sum_{n=0}^{\infty} \frac{z^{n(n+1)}}{n^n}?$$
How does the series behave on the boundary of the disk where it converges?
4. Identify all the poles, zeros, removable singularities, and essential singularities of the function
$$f(z) = \frac{1}{z} e^{z+\frac{1}{z}}$$
Be sure to consider the behavior as $z \to \infty$. What is the residue of f at 0, i.e., the value of the integral
$$\frac{1}{2\pi i} \int_{|z|=1} f(z)dz?$$
Express the answer as an infinite series.
5. Assume $u : \mathbb{R}^3 \to \mathbb{R}$ is a smooth function that satisfies
$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} > 0.$$
Show that u cannot assume a local maximum at any point $(x_0.y_0, z_0) \in \mathbb{R}^3$.
6. Let $\mathbf{x} : [0, \infty) \to \mathbb{R}^3$ denote a solution of the ODE
$$\dot{\mathbf{x}}(t) = \boldsymbol{c} \times \mathbf{x}(t), \boldsymbol{x}(0) = \boldsymbol{x}_0.$$
Here $\boldsymbol{c}$ is a fixed unit vector, $\times$ is the standard cross product on $\mathbb{R}^3$. Show that the trajectory of $\mathbf{x}$ lies on a circle in $\mathbb{R}^3$, and give an equation for the circle.

7. Show that the linear system

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 2 \\ 1 & 3 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

is not solvable for arbitrary $(a, b, c)^t$. Find a linear condition that insures solvability.

8. You are waiting at a bus stop, where the arrival times of buses are Poisson processes. Bus A has an arrival rate of μ, Bus B has an arrival rate of ν, and they arrive independently. What is the probability that Bus A arrives before Bus B?