TYPE-B Q,T-CATALAN NUMBERS AND BINOMIAL COEFFICIENTS

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Abstract

It would be very desirable to have a combinatorial description for $B_n(q,t)$, the bigraded characters of the type B analogue of the q,t-Catalan polynomial. It is known that $B_n(1,t) = \sum_{\pi \in \mathcal{E}_n} t^{\text{area}}$, where $\mathcal{E}_n$ is the set of "shifted" lattice paths consisting of unit North and East steps. In this thesis, we study a broader class of objects by increase the length of the bottom of the shifted paths to k, in the hope of finding a uniform combinatorial description for them all.

We here state the assumptions for this broader class $B_{n,k}(q,t)$ and give a candidate description of $B_{n,k}(q,t)$ as a positive linear combination of sl_2-strings for $n = 2,3$, and a combinatorial description for $n = 2$. For general n,k, we then state and proof a recurrence relation for $B_{n,k}(q,1/q)$ as a positive linear combination of sl_2-strings, and finally give a recurrence relation involving Catalan numbers of type A.

Contents

1 Introduction to q,t-Catalan numbers

The (type A) q, t-Catalan polynomial $C_n(q, t)$ has three main interpretations. In this section we will introduce the original analytic definition given by Garsia and Haiman, the generalize q, t - Catalan numbers as a bigraded Hilbert Series, and the pure combinatorial description proved by Garsia and Haglund. Then we will introduce the q, t-Fuss-Catalan numbers for complex reflection groups presented by Stump, and a few results on the specialization of the type B q, t-Catalan numbers $B_n(q, t)$.

1.1 Catalan number and two q-analogues

Let $C_n = \frac{1}{n+1}\binom{2n}{n}$ denote the n-th Catalan number. One of the objects counted by the Catalan numbers is the number of Dyck paths in $\mathcal{D}_n$, defined as follows (a list of other objects counted by the Catalan numbers could be found in [13]):

Definition 1.1. A *Dyck path* is a sequence of North $N(0, 1)$ and East $E(1, 0)$ steps in the first quadrant of the xy-plane, starting at the origin $(0, 0)$, ending at (n, n), and never go below the diagonal $y = x$. We let $\mathcal{D}_n$ denote the set of all such path.

There are two natural q-analogues of C_n. The first was studied by MacMahon[12]. Given $\lambda \in \mathcal{D}_n$, let $\sigma(\lambda)$ be the element of a linear list of the multiset $\{0^n 1^n\}$ resulting from the following algorithm: (1) Initialize σ to the empty string. (2) Start at $(0, 0)$, move along λ and add a 0 to the end of $\sigma(\lambda)$ every time a N step is encountered, and add a 1 to the end of $\sigma(\lambda)$ every time an E step is encountered. Then first q-analogues of C_n was given as following:

Theorem 1.2. *(MacMahon)*

$$\sum_{\lambda \in \mathcal{D}_n} q^{\mathrm{maj}(\sigma(\lambda))} = \frac{1}{[n+1]_q} \begin{bmatrix} 2n \\ n \end{bmatrix}_q,$$

where maj is the major-index statistic defined by $\mathrm{maj}(\sigma) = \sum_{\sigma_i > \sigma_{i+1}}^{i} i$, *and as is customary* $[k]_q = (1-q^n)/(1-q)$, $[k]_q! = [1]_q[2]_q \cdots [k]_q$, $\left[{n \atop k} \right]_q = [n]_q!/([k]_q![n-k]_q!)$.

The second natural q-analogue of C_n was studied by Carlitz and Riordan[4]. Given $\lambda \in \mathcal{D}_n$, let $a_i(\lambda)$ denote the number of complete squares, in the i-th row from the bottom of λ, which are to the right of λ and to the left of the line $y = x$. We refer to $a_i(\lambda)$ as the length of the i-th row of λ.

Set

$$\mathrm{area}(\lambda) = \sum_i a_i(\lambda)$$

be the *area statistic* of λ(see Figure 1 as an example).

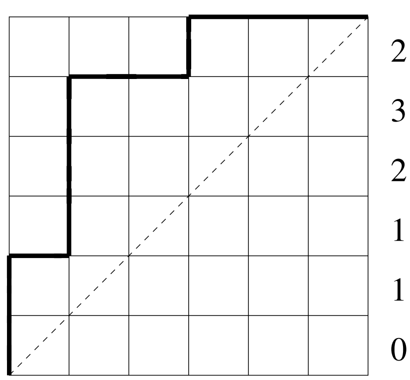


Figure 1: A Dyck path, with row lengths on the right. The area statistic is $1+1+2+3+2 = 9$.

Then first q-analogues of C_n was given as following:

Theorem 1.3. *(Carlitz & Riordan)*

Define $C_n(q) = \sum_{\lambda \in \mathcal{D}_n} q^{\mathrm{area}(\lambda)}$, *then*

$$C_n(1) = C_n = \frac{1}{n+1}\binom{2n}{n},$$

and

$$C_n(q) = \sum_{k=1}^{n} q^{k-1} C_k(q) C_{n-k}(q), \quad n \geq 1.$$

1.2 Analytic definition of (type A) q, t Catalan polynomial by Garsia and Haiman

The q,t-Catalan polynomial $C_n(q,t)$ was originally introduced in a paper by A.M. Garsia and M.Haiman in [7] analytically (see (1.1) below), as a sum over rational functions of q, t, which arise when applying the Macdonald polynomial ∇ operator to the nth elementary symmetric function. This analytic definition was partially motivated by an algebraic description suggested by Haiman [11] as the bigraded sign character of the space of diagonal harmonics.

Definition 1.4. (Garsia & Haiman)

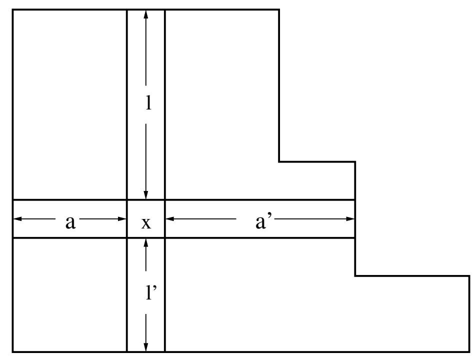


Figure 2: Arm, leg, co-arm and co-leg.

The q,t-Catalan sequence is defined by setting

$$C_n(q,t) = \sum_{\mu \vdash n} \frac{t^{2\sum_x l(x)} q^{2\sum_x a(x)} (1-t)(1-q) \prod_{x \neq (0,0)} (1 - q^{a'(x)} t^{l'(x)}) \sum_x q^{a'(x)} t^{l'(x)}}{\prod_x (q^{a(x)} - t^{l(x)+1})(t^{l(x)} - q^{a(x)+1})}. \tag{1.1}$$

where the sum is over all partitions μ of n, and all products and sums in the μ^{th} summand are over the cells x of μ. For a given cell x, in the Ferrers diagram of μ, the leg $l(x)$, the arm $a(x)$, the co-leg $l'(x)$, and the co-arm $a'(x)$ of x are defined to be respectively the numbers of squares above, to the right of, below, and to the left of x, with the diagram oriented in the French manner as shown in Figure 2.

This $C_n(q,t)$ is a bivariate "q-analog" of the familiar Catalan numbers $C_n = \frac{1}{n+1}\binom{2n}{n}$. Making use of the theory of Macdonald polynomials, the paper established that the specialization $C_n(1,q)$ and $q^{\binom{n}{2}} C_n(q,1/q)$ reduce to well-known q-analogues of the Catalan numbers, state as follows, respectively.

Proposition 1.5. $C_n(q) = C_n(1,q) = C_n(q,1)$ *reduces to the Carlitz-Riodan [4] q-*

Catalan numbers:

$$C_0(q) = 1$$
$$C_n(q) = \sum_{k=0}^{n-1} q^k C_k(q) C_{n-1-k}(q) = \sum_{\lambda \in \mathcal{D}_n} q^{\mathrm{area}(\lambda)},$$

which q-counts Dyck paths by area.

Proposition 1.6. $D_n(q) = q^{\binom{n}{2}} C_n(q, 1/q)$ *reduces to MacMahon[12] q-Catalan numbers:*

$$D_n(q) = \frac{1}{[n+1]_q} \begin{bmatrix} 2n \\ n \end{bmatrix}_q = \sum_{\lambda \in \mathcal{D}_n} q^{\mathrm{maj}(\sigma(\lambda))},$$

which q-counts Dyck words by the major index.

1.3 generalized q, t-Catalan numbers as a bigraded Hilbert Series

In [10], M. Haiman introduced the following definition as a bigraded Hilbert Series.

Beginning with the polynomial ring

$$\mathbb{C}[X_n, Y_n] := \mathbb{C}[x_1, y_1, \cdots, x_n, y_n],$$

recall the definition of bigraded Hilbert series:

Given any subspace $W \subseteq \mathbb{C}[X_n, Y_n]$, the bigraded Hilbert series of W is defined as

$$\mathcal{H}(W;q,t) = \sum_{i,j\geq 0} t^i q^j \dim(W^{(i,j)}),$$

where the subspaces $W^{(i,j)}$ consist of those elements of W of bi-homogeneous degree i in the x variables and j in the y variables, so $W = \oplus_{i,j\geq 0} W^{(i,j)}$.

let the symmetric group $\mathcal{S}_n$, which is the reflection group of type A_{n-1}, act diagonally by permuting the coordinates in x and y simultaneously amongst themselves, i.e.

$$\sigma p(x_1, y_1, \cdots, x_n, y_n) = p(x_{\sigma(1)}, y_{\sigma(2)}, \cdots, x_{\sigma(n)}, y_{\sigma(n)}).$$

A polynomial $p \in \mathbb{C}[X_n, Y_n]$ is alternating, or an alternate, if

$$\sigma(p) = \operatorname{sgn}(p), \quad \forall \sigma \in \mathcal{S}_n.$$

Let W^ϵ be the subspace of alternating elements in W, and

$$\mathcal{H}(W^\epsilon;q,t) = \sum_{i,j\geq 0} t^i q^j \dim(W^{\epsilon(i,j)}).$$

Now let I be the ideal generated by all S_n-invariant polynomials without constant term: $I = \left\langle \sum_{i=1}^n x_i^h y_i^k, \forall h+k>0 \right\rangle$, consider the quotient ring

$$DR_n = \mathbb{C}[X_n, Y_n]/I.$$

Also define the space of diagonal harmonics DH_n by

$$N_n = \{f(X_n, Y_n) : p(\partial X_n, \partial Y_n)f(X_n, Y_n) = 0, \forall p(X_n, Y_n) \in I\},$$

where $p(\partial X_n, \partial Y_n)$ denotes the differential operator obtained by substituting for the variables $x_1, y_1, \cdots, x_n, y_n$ the corresponding partial derivative operators $\partial_{x_1}, \partial_{y_1}, \cdots,$ $\partial_{x_n}, \partial_{y_n}$.

Many of the properties of DH_n and RH_n carry over to two sets of variables. For example DH_n is a finite dimensional vector space which is isomorphic to DR_n. Among several conjectures of a combinatorial nature concerning DH_n is that its subspace DH_n^ϵ of S_n-alternating elements – that is, its isotypic component corresponding to the sign character ϵ of S – has dimension equal to the Catalan number C_n.

Definition 1.7. (Haiman)

Taking into account the grading, Haiman defined a Hilbert polynomial

$$D_n(t, q) = \mathcal{H}(DH_n^\epsilon; q, t) = \sum_{h,k\geq 0} t^h q^k \dim(DH_n^\epsilon)_{h,k}.$$

Haiman later showed that the rational function $C_n(q, t)$ defined above is in fact equals to the bigraded Hilbert series defined above:

Theorem 1.8. *(Haiman)*

$$C_n(q, t) = D_n(q, t) = \mathcal{H}(DH_n^\epsilon; q, t).$$

1.4 Pure combinatorics definitions

A pure combinatorial description was later conjectured by Haglund in 2000[8] after a pro-longed study of tables of $C_n(q,t)$. It was then proved by Garsia and Haglund [5] [6].

This combinatorial formula for $C_n(q,t)$ involves a new statistic on Dyck path called bounce.

Definition 1.9. Given $\lambda \in \mathcal{D}_n$, define the *bounce path* of λ to be the path described by the following algorithm: (1)Start at $(0,0)$ and travel North along λ. (2) When encounter the beginning of an E step, turn East and travel straight. (3) When hit the diagonal $y = x$. Then turn North and travel straight. (4) Continue in this way until you arrive at (n,n).

The "bouncing ball" will strike the diagonal at places $(0,0), (j_1, j_1), (j_2, j_2), \cdots, (j_b - 1, j_b - 1), (j_b, j_b) = (n,n)$. We define the bounce statistic bounce(λ) to be the sum

$$\text{bounce}(\lambda) = \sum_{i=1}^{b-1} n - j_i.$$

An example is shown in Figure 3.

The following theorem is proved by Garsia and Haglund [5] [6]:

Theorem 1.10. *(Garsia & Haglund)*

$$C_n(q,t) = \sum_{\lambda \in \mathcal{D}_n} q^{bounce(\lambda)} t^{area(\lambda)}.$$

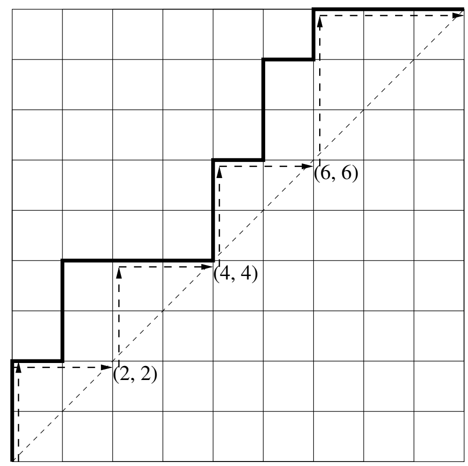


Figure 3: The bounce path (dotted line) of a Dyck path (solid) line. The bounce statistic equals $9-2+9-4+9-6=7+5+3=15$.

There is another pair of statistics for the q,t-Catalan discovered by M. Haiman. It involves pairing area with a different statistic we call "dinv", for "diagonal inversion" or "d-inversion".

Definition 1.11. Let $\lambda \in \mathcal{D}_n$. Let

$$\begin{aligned}\operatorname{dinv}(\lambda) = \ & |\{(i,j) : 1 \le i < j \le n, a_i = a_j\}| \\ & + |\{(i,j) : 1 \le i < j \le n, a_i = a_j + 1\}|\end{aligned}$$

be the *"diagonal inversion"* or *"d-inversion"* statistics, where a_i the length of the i-th row from the bottom.

In words, $\operatorname{dinv}(\lambda)$ is the number of pairs of rows of λ of the same length, or which differ by one in length, with the longer row below the shorter, as shown in Figure 4.

Then we have the following theorem and corollary:

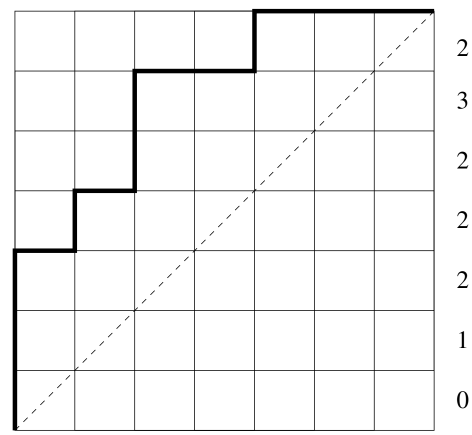


Figure 4: the inversion pairs (i, j) are $(3,4),(3,5),(3,7),(4,5),(4,7),(5,7)$ (corresponding to pairs of rows of the same length) and $(6,7)$ (corresponding to rows which differ by one in length), thus dinv $= 7$.

Theorem 1.12.

$$\sum_{\lambda \in \mathcal{D}_n} q^{dinv(\lambda)} t^{area(\lambda)} = \sum_{\lambda \in \mathcal{D}_n} q^{bounce(\lambda)} t^{area(\lambda)}.$$

Corollary 1.13.

$$C_n(q,t) = \sum_{\lambda \in \mathcal{D}_n} q^{dinv(\lambda)} t^{area(\lambda)}.$$

1.5 q,t-Fuss-Catalan numbers for complex reflection groups

As introduced in section 1.3, $C_n(q,t)$ can be defined as a bigraded Hilbert series of a module associated to the symmetric group $\mathcal{S}_n$. Stump[14] generalized this construction to finite complex reflection groups and exhibit some nice conjectured algebraic and combinatorial properties of these polynomials in q and t.

Generalize the concept for polynomials to be alternating to any finite complex reflection group in the following way: let V be an n-dimensional complex vector space and let $W \subset \mathrm{GL}(V)$ be a finite complex reflection group acting on V. Definitions on complex reflection groups could be found in [3].

Define the diagonal action of W on $\mathbb{C}[X_n, Y_n]$ by "doubling up" the contragredient action $\omega(\rho) := \rho \circ \omega^{-1}$ of W on $V^* = \mathrm{Hom}(V, \mathbb{C})$ diagonally.

Let W be a complex reflection group acting on a complex vector space of dimension n. We call a polynomial $p \in \mathbb{C}[X_n, Y_n]$ alternating if

$$\det(\omega)\omega(p) = p, \quad \forall \omega \in W.$$

Let $I \trianglelefteq \mathbb{C}[X_n, Y_n]$ be the ideal generated by all alternating polynomials and define the W-module $M^{(m)} := I^m / < X_n, Y_n > I^m$.

Definition 1.14. (Stump)

The q, t-Fuß-Catalan numbers associated to W are defined as

$$\mathrm{Cat}^{(m)}(W, q, t) := \mathcal{H}(M^{(m)}; q, t) = \sum_{i,j>0} \dim(M_{ij}) q^i t^j.$$

Remark 1.15. For W being the complex reflection group of type A_{n-1} - which is the symmetric group $\mathcal{S}_n$ - the definition of alternating polynomials reduces to the case given in section1.3, and the definition of q, t-Fuß-Catalan numbers associated to $W = \mathcal{S}_n$ is reduced to definition 1.7. Later on we will call this $W = S_n$ case $C_n(q, t)$ the type-A q, t-Catalan numbers.

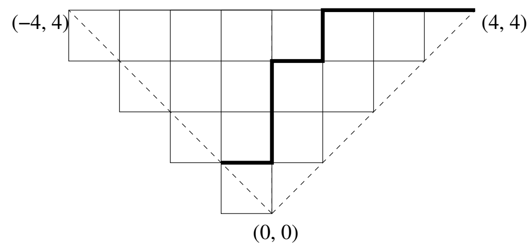


Figure 5: A shifted Dyck path in $\mathcal{E}_4$ with area = 6

1.6 Type B analogue of q, t-Fuß-Catalan numbers

Define $B_n(q,t) = \mathrm{Cat}^{(1)}(W_{B_n}, q, 1)$ to be the Type B analogue of q, t-Fuß-Catalan numbers. It would be very desirable to have a combinatorial description for the bigraded characters of the type B analogue $B_n(q,t)$, as well as for other root systems. There is currently no known way to define them analytically or combinatorially. There are, however, some studies on the specialization $B_n(1,q)$ and $q^{n^2} B_n(q, 1/q)$.

Stump has conjectured in [14] and that $B_n(1,q)$ q-counts the area statistic for Catalan paths of type B and establish an analogous recurrence.

Definition 1.16. A type B Catalan path(or a "shifted Dyck path") of length n, denoted as $\mathcal{E}_n$, is a lattice paths of $2n$ steps, either north or east, that starts at some point on the anti-diagonal $y = -x$, ends at (n,n) and stays above the diagonal $x = y$. For such a path λ, we define area(λ) to be the number of boxes in the region confines by the path, the diagonal $y = x$ and the anti-diagonal $y = -x$, not counting the halfboxes at the diagonal $y = x$ but counting the halfboxes at the anti-diagonal $y = -x$.

If Conjecture 1.17 is correct, the following propositions are comparable to proposition 1.5.

Conjecture 1.17. *(Stump)*

$$B_n(1,q) = B_n(q,1) = \sum_{\lambda \in \mathcal{E}_n} q^{\mathrm{area}(\lambda)}.$$

Proposition 1.18. *(Stump)*

$B_n(1,q)$ satisfy the following recurrence involving Catalan numbers of type A:

$$B_n(1,q) = C_n(1,q) + \sum_{k=0}^{n-1} q^{2k+1} B_k(1,q) C_{n-k}(1,q).$$

Also, similar to proposition 1.6, we have the following conjecture by Haiman[9]:

Conjecture 1.19. *(Haiman)*

$$q^{n^2} B_n(q, 1/q) = \begin{bmatrix} 2n \\ n \end{bmatrix}_{q^2}$$

1.7 Schur polynomials and sl_2-strings

Recall the definition of Schur polynomials:

Definition 1.20. Given a partition $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_n)$, where $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$, and each λ_j is a non-negative integer, the Schur polynomials are defined as the ratio

$$s_\lambda(x_1, x_2, \ldots, x_n) = \frac{a_{(\lambda_1+n-1, \lambda_2+n-2, \ldots, \lambda_n+0)}(x_1, x_2, \ldots, x_n)}{a_{(n-1, n-2, \ldots, 0)}(x_1, x_2, \ldots, x_n)}.$$

where

$$a_{(\lambda_1+n-1,\ldots,\lambda_n)}(x_1,x_2,\ldots,x_n) = \det\begin{bmatrix} x_1^{\lambda_1+n-1} & x_2^{\lambda_1+n-1} & \ldots & x_n^{\lambda_1+n-1} \\ x_1^{\lambda_2+n-2} & x_2^{\lambda_2+n-2} & \ldots & x_n^{\lambda_2+n-2} \\ \vdots & \vdots & \ddots & \vdots \\ x_1^{\lambda_n} & x_2^{\lambda_n} & \ldots & x_n^{\lambda_n} \end{bmatrix}.$$

Specifically, in the case of $n = 2$, we have

$$\begin{aligned} s_{\lambda_1,\lambda_2}(q,t) &= (a_{\lambda_1+1,\lambda_2(q,t)})/(a_{1,0}(q,t)) \\ &= ((q^{\lambda_1+1}t^{\lambda_2} - q^{\lambda_2}t^{\lambda_1+1})/(q-t) \\ &= (qt)^{\lambda_2}(q^{\lambda_1-\lambda_2+1} - t^{\lambda_1-\lambda_2+1})/(q-t) \end{aligned}$$

Note that such a Schur polynomial can be written in the form $(qt)^m(q^k - t^k)/(q-t)$. We say a polynomial of this form is an sl_2-string. To simplify, we use the following notation given by [1]: Let

$$[k]_{q,t} := (q^k - t^k)/(q-t).$$

Remark 1.21. Note that under this notation,

$$[k]_{q,1} = [k]_{1,q} = [k]_q$$

and

$$s_{\lambda_1,\lambda_2}(q,t) = (qt)^{\lambda_2}[\lambda_1 - \lambda_2 + 1]_{q,t}.$$

As a corollary of the result stated in [2], we have $B_n(q,t)$ is a positive coefficient linear combination of Schur polynomials. Hence we have the following proposition:

Proposition 1.22. *$B_n(q,t)$ is a positive coefficient linear combination of sl_2-strings.*

2 Set-up: Problem statement and basic assumptions to $B_{n,k}(q,t)$

In this section we will state the motivation and formally define the assumptions for the broader class of polynomials $B_{n,k}(q,t)$ that we will study.

Recall the open problem that is stated by Stump in [14]:

Open Problem 2.1. *Are there statistics qstat and tstat on objects counted by $Cat^{(m)}(W)$ which generalize area and bounce on Catalan paths $\mathcal{D}_n^m$ such that*

$$Cat^{(m)}(W,q,t) = \sum_{\lambda} q^{qstat(\lambda)} t^{tstat(\lambda)}?$$

We still have no candidate for the type B version of dinv (or perhaps a type B version of bounce) to match with area to generate $B_n(q,t) = \sum_{\lambda \in \mathcal{E}_n} q^{\mathrm{qstat}(\lambda)} t^{\mathrm{area}(\lambda)}$.

2.1 Shifted Dyck path with base k and area statistics

By observation, if you increase the length of the bottom of the shifted paths to k, as shown in Figure 6, then the total number of paths seems to be the binomial coefficient $\binom{2n+k-1}{n}$. It might make sense to study this broader class of objects, in the hope of finding a uniform combinatorial description for them all, perhaps via an undiscovered recurrence relation. Formally, we define the shifted Dyck path with base k and the area statistics for such a path as follows:

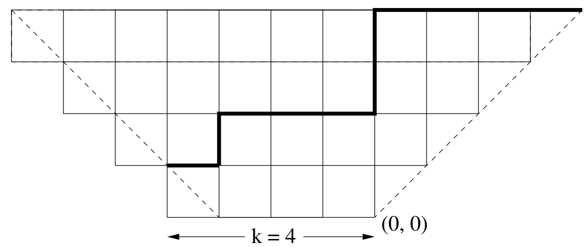


Figure 6: A shifted Dyck path with base $k = 4$, length $n = 4$, area = 13

Definition 2.2. A "shifted Dyck path" with base k of length n, denoted as $\mathcal{E}_{n,k}$, is a lattice paths of $2n + k - 1$ steps, either north or east, that starts at some point on the anti-diagonal $y = 1 - k - x$, ends at (n, n) and stays above the diagonal $x = y$. For such a path λ, we define area(λ) to be the number of boxes in the region confines by the path, the diagonal $y = x$ and the anti-diagonal $y = 1 - k - x$, not counting the halfboxes at the diagonal $y = x$ but counting the halfboxes at the anti-diagonal $y = 1 - k - x$.

2.2 q, t polynomials $B_{n,k}(q, t)$

Our goal following would be trying to find q, t polynomials $B_{n,k}(q, t)$ for the shifted paths with base k, where $B_{n,1}(q, t) = B_n(q, t)$.

We will try to find a candidate description of $B_{n,k}(q, t)$ as a positive linear combination of sl_2-strings, based on the following two assumptions on the specialization $B_{n,k}(1, t)$ and $B_{n,k}(q, 1/q)$:

Assumption 2.3.

$$q^{n(n+k-1)}B_{n,k}(q,1/q) = \begin{bmatrix} 2n+k-1 \\ n \end{bmatrix}_{q^2}. \tag{2.1}$$

Assumption 2.4.

$$B_{n,k}(1,t) = \sum_{\pi \in \varepsilon_{n,k}} t^{area}. \tag{2.2}$$

Remark 2.5. Suppose we want a positive linear combination of sl_2-strings:

$$B_{n,k}(q,t) = \sum_i (qt)^{m_i}[l_i]_{q,t}.$$

Note that

$$\begin{aligned} (q \cdot 1/q)^m [l]_{q,1/q} &= [l]_{q,1/q} \\ &= (q^l - q^{-l})/(q - q^{-1}) \;\cdot \\ &= q^{-l+1}[l]_{q^2} \end{aligned}$$

and

$$(1 \cdot t)^m [l]_{1,t} = t^m [l]_t.$$

We can see that $B_{n,k}(q,1/q)$ gives which terms of l_i there are in the sum of $(q,t)^{m_i}[l_i]_{q,t}$, and $B_{n,k}(1,t)$ gives us an idea of the "coefficients" $(qt)^{m_i}$ for each $[l_i]_{q,t}$.

3 $n = 2$: $B_{2,k}(q,t)$ as a positive linear combination of sl_2-strings

In this section we will calculate $B_{2,k}(q,1/q)$ and $B_{2,k}(1,t)$ according to the assumptions, and write them as positive linear combination of sl_2-strings, respectively. Then we will give a candidate description of $B_{3,k}(q,t)$ as a positive linear combination of sl_2-strings, and give a candidate combinatorics description for this proposed $B_{2,k}(q,t)$.

3.1 Calculation of $B_{2,k}(q,1/q)$

Note that by (2.1),

$$q^{2k+2}B_{2,k}(q,1/q) = \begin{bmatrix} k+3 \\ 2 \end{bmatrix}_{q^2}.$$

Hence

$$\begin{aligned} B_{2,k}(q,1/q) &= (\left[\begin{smallmatrix} k+3 \\ 2 \end{smallmatrix}\right]_{q^2})/(q^{2k+2}) \\ &= (1-(q^2)^{k+3})(1-(q^2)^{k+2})/((1-q^2)(1-(q^2)^2)q^{2k+2}) \end{aligned}$$

- When k is odd,

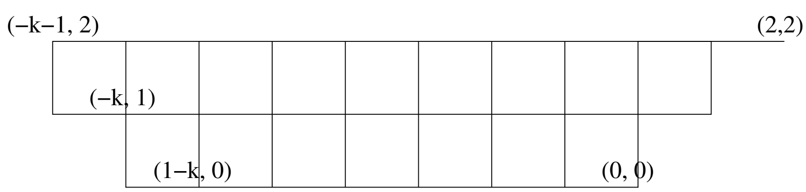
$$\begin{aligned} B_{2,k}(q,1/q) &= q^{-2k-2}[k+2]_{q^2}[(k+3)/2]_{q^4} \\ &= q^{-2k-2}([2k+3]_{q^2} + q^4[2k-1]_{q^2} + \cdots + q^{2k}[3]_{q^2}) \\ &= q^{-(2k+3)+1}[2k+3]_{q^2} + q^{-(2k-1)+1}[2k-1]_{q^2} + \cdots + q^{-3+1}[3]_{q^2} \\ &= [2k+3]_{q,1/q} + [2k-1]_{q,1/q} + \cdots + [1]_{q,1/q}. \end{aligned} \tag{3.1}$$

- When k is even,

$$\begin{aligned} B_{2,k}(q,1/q) &= q^{-2k-2}[k+3]_{q^2}[(k+2)/2]_{q^4} \\ &= q^{-2k-2}([2k+3]_{q^2} + q^4[2k-1]_{q^2} + \cdots + q^{2k+2}[1]_{q^2}) \\ &= q^{-(2k+3)+1}[2k+3]_{q^2} + q^{-(2k-1)+1}[2k-1]_{q^2} + \cdots + q^{-1+1}[1]_{q^2} \\ &= [2k+3]_{q,1/q} + [2k-1]_{q,1/q} + \cdots + [3]_{q,1/q}. \end{aligned} \tag{3.2}$$

3.2 Calculation of $B_{2,k}(1,t)$

The shifted Dyck path can start from $(1-k,0)$, $(-k,1)$ or $(-k-1,2)$.



- case 1: starting from $(-k-1,2)$. Apparently, the only path here is taking every step East.

$$\sum_{\pi\in\mathcal{E}_{2,k},\text{start point }=(-k-1,2)} t^{\text{area}} = t^{2k+2}.$$

- case 2: starting from $(-k,1)$. There are $k+1$ choices to take the step North.

$$\sum_{\pi\in\mathcal{E}_{2,k},\text{start point }=(-k,1)} t^{\text{area}} = t^k(1+t+t^2+\cdots+t^{k+1}) = t^k[k+2]_t.$$

- case 3: starting from $(1-k,2)$. There are k choices to take the first step North, and then the situation reduces to one in case 2.

$$\sum_{\pi\in\mathcal{E}_{2,k},\text{start point }=(1-k,0)} t^{\text{area}} = t^{k-1}[k+1]_t + t^{k-2}[k]_t + \cdots + t^0[2]_t.$$

From the area statistics, we can conclude that

$$\begin{aligned}
B_{2,k}(1,t) = \quad & \textstyle\sum_{\pi\in\mathcal{E}_{2,k}} t^{\text{area}} \\
= \quad & t^{2k+2} + t^k[k+2]_t + t^{k-1}[k+1]_t + t^{k-2}[k]_t + \cdots + t^0[2]_t \\
= \quad & t^{2k+2} + t^k[k+2]_t + (t^{k-1} + t^k[k]_t) + (t^{k-2} + t^{k-1}[k-1]_t) \\
& + \cdots + (t^0 + t^1[1]_t). \\
= \quad & [2k+3]_t + t(t^{k-1}[k]_t + t^{k-2}[k-1]_t + \cdots t^0[1]_t) \\
= \quad & [2k+3]_t + t[2k-1]_t + t^3(t^{k-3}[k-2]_t + \cdots t^0[1]_t) \\
= \quad & \begin{cases} [2k+3]_t + t[2k-1]_t + t^3[2k-5]_t \cdots + t^k[1]_t & 2 \nmid k \\ [2k+3]_t + t[2k-1]_t + t^3[2k-5]_t \cdots + t^{k-1}[3]_t & 2 \mid k \end{cases}.
\end{aligned}$$

3.3 A candidate description of $B_{2,k}(q,t)$ as a sum of sl_2-strings

By the results of $B_{2,k}(1,t)$ and $B_{q,1/q}$, there is a unique way of writing $N_{2,k}(q,t)$ as a sum of sl_2-strings:

$$B_{2,k}(q,t) = \begin{cases} [2k+3]_{q,t} & +(qt)[2k-1]_{q,t} + (qt)^3[2k-5]_{q,t} \\ & +(qt)^5[2k-9]_{q,t} + \cdots + (qt)^k[1]_{q,t} \quad 2 \nmid k \\ [2k+3]_{q,t} & +(qt)[2k-1]_{q,t} + (qt)^3[2k-5]_{q,t} \\ & +(qt)^5[2k-9]_{q,t} + \cdots + (qt)^{k-1}[3]_{q,t} \quad 2 \mid k \end{cases} .$$

3.4 A candidate combinatorics description for this proposed $B_{2,k}(q,t)$

To try to give a pure combinatorics description to this $B_{2,k}(q,t)$, we want to decide the power of each $q^{\text{stat1}}t^{\text{stat2}}$ of each shifted Dyck path with base $k=4$ in a reasonable way.

Suppose we want the second power to be the area statistic. We consider each group of paths in the cases stated in the calculation of $B_{2,k}(1,t)$.

	$t^k[k+2]_t$	$t^{k-1}[k+1]_t$	$t^{k-2}[k]_t$	$\cdots\cdots$	$t^1[3]_t$	$t^0[2]_t$
t^{2k+2}	1	0	0	$\cdots$	0	0
t^{2k+1}	1	0	0	$\cdots$	0	0
t^{2k}	1	0	0	$\cdots$	0	0
$t^{2k}-1$	1	1	0	$\cdots$	0	0
$t^{2k}-2$	1	1	0	$\cdots$	0	0
$t^{2k}-3$	1	1	1	$\cdots$	0	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
t^k	1	1	1	$\cdots$	0	0
t^{k-1}	0	1	1	$\cdots$	0	0
t^{k-2}	0	0	1	$\cdots$	0	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
t^3	0	0	0	$\cdots$	1	0
t^2	0	0	0	$\cdots$	1	0
t^1	0	0	0	$\cdots$	1	1
t^0	0	0	0	$\cdots$	0	1

As shown in the chart above, it is natural to see that when you add up all 1 the terms in the first column, and then going diagonal taking the last item in every next columns till the last column(which are those marked red), we will get $[2k+3]_t$.

Visually, we call that the "first layer" since those are the terms in the left and bottom most in the chart. Similarly, we get the"second layer" as the terms marked blue. Those terms add up to $t[2k-1]_k$. Continue this grouping process until we hit the inner-most layer.

In comparison to the candidate $B_{2,k}(q,t)$, we let any path corresponds to the term t^i in the first layer $[2k+3]_t$ to be the corresponding term in $[2k+3]_{q,t}$, which is

$$q^{2k+2-i}t^i$$

. Similarly, any path corresponds to the term $t^{2j-3}t^i$ in the j-th layer $t^{2j-3}[2k-(4j-7)]_t$ to be the corresponding term in $(qt)^{2j-3}[2k-(4j-7)]_{q,t}$, which is

$$(qt)^{2j-3}t^iq^{2k-(4j-7)-1-i} = q^{2k-2j-i+3}t^{2j+i-3}$$

.

Note that $(2k+2-i)+i=i$ and $(2k-2j-i+3)+(2j+i-3)=2k$, we have

$$\begin{cases} q^{2k+2-\text{area}}t^{\text{area}} & \text{path in "first layer"} \\ q^{2k-\text{area}}t^{\text{area}} & \text{otherwise} \end{cases}.$$

An example for $k=4$ is as follows:

	$t^4[6]_t$	$t^3[5]_t$	$t^2[4]_t$	$t^1[3]_t$	$t^0[2]_t$
t^{10}	t^{10}				
t^9	qt^9				
t^8	q^2t^8				
t^7	q^3t^7	$(qt)t^6 = qt^7$			
t^6	q^4t^6	$(qt)qt^5 = q^2t^6$			
t^5	q^5t^5	$(qt)q^2t^4 = q^3t^5$	$(qt)^3t^2 = q^3t^5$		
t^4	q^6t^4	$(qt)q^3t^3 = q^4t^4$	$(qt)^3qt = q^4t^4$		
t^3		q^7t^3	$(qt)q^4t^2 = q^5t^3$	$(qt)^3q^2 = q^5t^3$	
t^2			q^8t^2	$(qt)q^5t = q^6t^2$	
t				q^9t	$(qt)q^6 = q^7t$
t^0					q^{10}

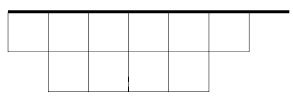
Now we use the language of the shifted Dyke path to describe those paths "in the first layer". Those in the first column are the paths in case 1 and case 2, which are those starting from $(-k-1,2)$ and $(-k,1)$, and the last element of each following columns are those with the last two steps NE. (i.e. those paths that passes $(1,1)$.)

Hence

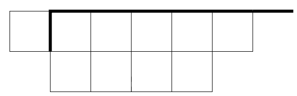
$$\begin{cases} q^{2k+2-\text{area}}t^{\text{area}} & \text{path passes } (-k-1,2),(-k,1) \text{ or } (1,1) \\ q^{2k-\text{area}}t^{\text{area}} & \text{otherwise} \end{cases}.$$

The example of $n = 4$ is shown as follows:

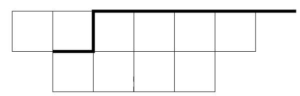
Example 3.1. For $n = 2, k = 4$, the count of each path is shown as follows:



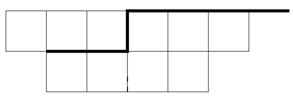
t^{10}



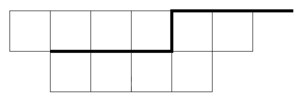
qt^9



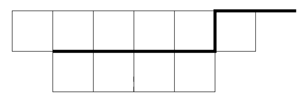
q^2t^8



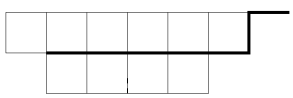
q^3t^7



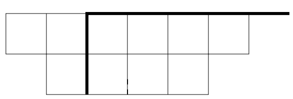
q^4t^6



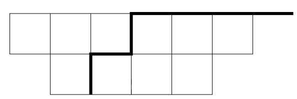
q^5t^5



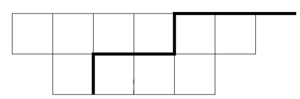
q^6t^4



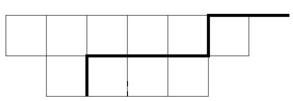
qt^7



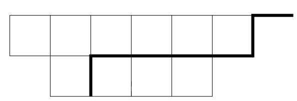
q^2t^6



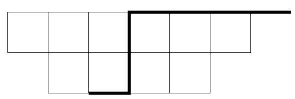
q^3t^5



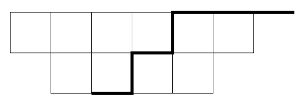
q^4t^4



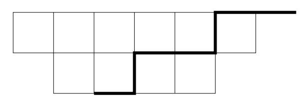
q^7t^3



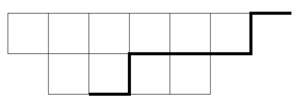
q^3t^5



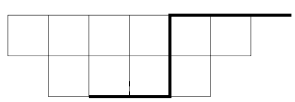
q^4t^4



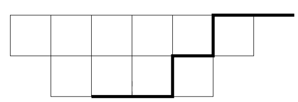
q^5t^3



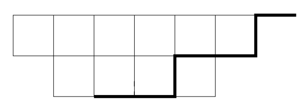
q^8t^2



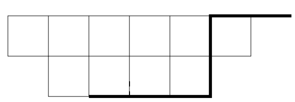
q^5t^3



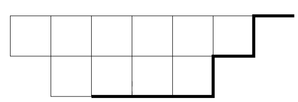
q^6t^2



q^9t



q^7t



q^{10}

4 $n = 3$: $B_{3,k}(q,t)$ as a positive linear combination of sl_2-strings

In this section we will state and prove the recurrence relation of $B_{3,k}(q, 1/q)$ and $B_{3,k}(1,t)$ written as a positive linear combination of sl_2-strings according to the assumptions, respectively. Before each formal proof, we will give an example a small k to help understanding. Finally, in the end of this section, we will give a candidate description of $B_{3,k}(q,t)$ as a positive linear combination of sl_2-strings.

4.1 Recurrence relation of $B_{3,k}(q, 1/q)$

Note that by (2.1),

$$q^{3k+6}B_{3,k}(q,1/q) = \begin{bmatrix} k+5 \\ 3 \end{bmatrix}_{q^2},$$

Hence

$$\begin{aligned} B_{3,k}(q,1/q) &= (\left[\begin{smallmatrix} k+5 \\ 3 \end{smallmatrix}\right]_{q^2})/(q^{3k+6}) \\ &= (1-(q^2)^{k+5})(1-(q^2)^{k+4})(1-(q^2)^{k+3})/((1-q^2)(1-(q^2)^2)(1-(q^2)^3)q^{3k+6}) \end{aligned}$$

We now simplify this formula and write it as a sum of sl_2-strings. The main result of this part is the recurrence relation stated as follows:

Theorem 4.1. *$B_{3,k}(q, 1/q)$ follows the following recursive form:*

(1) $B_{3,1}(q,1/q)$

$$B_{3,1}(q,1/q) = [10]_{q,1/q} + [6]_{q,1/q} + [4]_{q,1/q}.$$

(2) Suppose $B_{3,k} = \sum_i [x_i]_{q,1/q}$, *then*

$$B_{3,k+1}(q,1/q) = \sum_i [x_i + 3]_{q,1/q} + R_{3,k+1}(q,1/q)$$

where $R_{3,k+1}$ *is additional terms for* $k \geq 1$ *as follows:*

$$R_{3,k+1}(q,1/q) = \begin{cases} [1]_{q,1/q} + [5]_{q,1/q} + \cdots + [k+3]_{q,1/q} & k \equiv 2 \mod 4 \\ [6]_{q,1/q} + [10]_{q,1/q} + \cdots + [k+3]_{q,1/q} & k \equiv 3 \mod 4 \\ [3]_{q,1/q} + [7]_{q,1/q} + \cdots + [k+3]_{q,1/q} & k \equiv 0 \mod 4 \\ [4]_{q,1/q} + [8]_{q,1/q} + \cdots + [k+3]_{q,1/q} & k \equiv 1 \mod 4 \end{cases}$$

We will prove the theorem by induction. We will first give an example for $k = 4$ to help understand the frame of the proof, and then we will state the mathematical proof strictly.

4.1.1 Example: $B_{3,4}(q,1/q)$

Example 4.2. Suppose we have that the relation holds for $k=1$ through $k=3$, which gives

$$B_{3,1}(q,1/q) = [10]_{q,1/q} + [6]_{q,1/q} + [4]_{q,1/q}$$

$$B_{3,2}(q,1/q) = [13]_{q,1/q} + [9]_{q,1/q} + [7]_{q,1/q} + [5]_{q,1/q} + [1]_{q,1/q}$$

$$B_{3,3}(q,1/q) = [16]_{q,1/q} + [12]_{q,1/q} + [10]_{q,1/q} + [8]_{q,1/q} + [4]_{q,1/q} + [6]_{q,1/q}$$

Now we want to proof that the relation holds for $k=4$, i.e.

$$B_{3,4}(q,1/q) = [19]_{q,1/q}+[15]_{q,1/q}+[13]_{q,1/q}+[11]_{q,1/q}+[9]_{q,1/q}+[7]_{q,1/q}+[7]_{q,1/q}+[3]_{q,1/q}. \tag{4.1}$$

As a notation, we let the right hand side of (4.1) be $B'_{3,4}(q,1/q)$.

Note that we have

$$\begin{aligned} B_{3,k}(q,1/q) &= \tfrac{(1-(q^2)^{k+5})(1-(q^2)^{k+4})(1-(q^2)^{k+3})}{(1-q^2)(1-(q^2)^2)(1-(q^2)^3)q^{3k+6}} \\ &= \tfrac{(1-((1/q)^2)^{k+5})(1-((1/q)^2)^{k+4})(1-((1/q)^2)^{k+3})}{(1-(1/q)^2)(1-((1/q)^2)^2)(1-((1/q)^2)^3)(1/q)^{3k+6}} \end{aligned}$$

which means that both $B_{3,4}(q,1/q)$ and $B'_{3,4}(q,1/q)$ have the same coefficients for q^m and $(1/q)^m$ for any $m \in \mathbb{N}^*$. Hence we only need to prove that $B_{3,4}(q,1/q) -$

$B'_{3,4}(q, 1/q)$ has coefficient 0 for any q^m, $m \in \mathbb{N}^*$, which is equivalent to

$$(q^{18})B_{3,4}(q, 1/q) - q^{18}B'_{3,4}(q, 1/q) \equiv 0 \mod q^{18}.$$

Now we list every term of $q^{18}B'_{3,4}(q, 1/q)$ as follows:

	$q^{18}[19]_{q,\frac{1}{q}}$	$q^{18}[15]_{q,\frac{1}{q}}$	$q^{18}[13]_{q,\frac{1}{q}}$	$q^{18}[11]_{q,\frac{1}{q}}$	$q^{18}[9]_{q,\frac{1}{q}}$	$q^{18}[7]_{q,\frac{1}{q}}$	$q^{18}[7]_{q,\frac{1}{q}}$	$q^{18}[3]_{q,\frac{1}{q}}$
1	1							
q^2	1							
q^4	1	1						
q^6	1	1	1					
q^8	1	1	1	1				
q^{10}	1	1	1	1	1			
q^{12}	1	1	1	1	1	1	1	
q^{14}	1	1	1	1	1	1	1	
q^{16}	1	1	1	1	1	1	1	1
q^{18}	1	1	1	1	1	1	1	1
q^{20}	1	1	1	1	1	1	1	1
q^{22}	1	1	1	1	1	1	1	
q^{24}	1	1	1	1	1	1	1	
q^{26}	1	1	1	1	1			
q^{28}	1	1	1	1				
q^{30}	1	1	1					
q^{32}	1	1						
q^{34}	1							
q^{36}	1							

Also note that according to assumption, $q^{15}B_{3,3}(q,1/q)$ is as follows:

	$q^{15}[16]_{q,\frac{1}{q}}$	$q^{15}[12]_{q,\frac{1}{q}}$	$q^{15}[10]_{q,\frac{1}{q}}$	$q^{15}[8]_{q,\frac{1}{q}}$	$q^{15}[6]_{q,\frac{1}{q}}$	$q^{15}[4]_{q,\frac{1}{q}}$
1	1					
q^2	1					
q^4	1	1				
q^6	1	1	1			
q^8	1	1	1	1		
q^{10}	1	1	1	1	1	
q^{12}	1	1	1	1	1	1
q^{14}	1	1	1	1	1	1
q^{16}	1	1	1	1	1	1
q^{18}	1	1	1	1	1	1
q^{20}	1	1	1	1	1	
q^{22}	1	1	1	1		
q^{24}	1	1	1			
q^{26}	1	1				
q^{28}	1					
q^{30}	1					

As shown in the blue part of the graph above,

$$q^{15}[m]_{q,1/q} \equiv q^{18}[m+3]_{q,1/q} \mod q^{18}$$

for $m = 16, 12, 10, 8, 6, 4$. Hence we only need to prove that

$$q^{18}B_{3,4}(q,1/q) \equiv q^{15}B_{3,3}(q,1/q) + q^{18}R_{3,4}(q,1/q) \tag{4.2}$$

$$= q^{15}B_{3,3}(q,1/q) + q^{18}[7]_{q,1/q} + q^{18}[3]_{q,1/q} \mod q^{18}. \tag{4.3}$$

Note that

$$q^{18}B_{3,4}(q,1/q) = \frac{(1-q^{18})(1-q^{16})(1-q^{14})}{(1-q^2)(1-q^4)(1-q^6)}$$

$$q^{15}B_{3,3}(q,1/q) = \frac{(1-q^{16})(1-q^{14})(1-q^{12})}{(1-q^2)(1-q^4)(1-q^6)}$$

Hence

$$q^{18}B_{3,4}(q,1/q) - q^{15}B_{3,3}(q,1/q) = q^{12}\frac{(1-q^{16})(1-q^{14})}{(1-q^2)(1-q^4)}$$

From the previous calculation we know that

$$B_{2,5}(q,1/q) = \frac{(1-q^{16})(1-q^{14})}{(1-q^2)(1-q^4)q^{12}} = [13]_{q,1/q} + [9]_{q,1/q} + [5]_{q,1/q} + [1]_{q,1/q}$$

Hence

$$q^{18}B_{3,4}(q,1/q) - q^{15}B_{3,3}(q,1/q) = q^{12}(q^{12}[13]_{q,1/q} + q^{12}[9]_{q,1/q} + q^{12}[5]_{q,1/q} + q^{12}[1]_{q,1/q})$$

Also Note that

$$q^{18}R_{3,4}(q,1/q) = q^{12}(q^6[7]_{q,1/q} + q^6[3]_{q,1/q})$$

We have

$$\begin{aligned}
& q^{18}B_{3,4}(q,1/q) - q^{15}B_{3,3}(q,1/q) - R_{3,4}(q,1/q) \\
= \;& q^{12}(q^{12}[13]_{q,1/q} + q^{12}[9]_{q,1/q} + q^{12}[5]_{q,1/q} + q^{12}[1]_{q,1/q} - q^{6}[7]_{q,1/q} - q^{6}[3]_{q,1/q}) \\
= \;& q^{12}(q^{12}[13]_{q,1/q} - q^{6}[7]_{q,1/q}) + q^{12}(q^{12}[9]_{q,1/q} - q^{6}[3]_{q,1/q}) + q^{12}(q^{12}[5]_{q,1/q} + q^{12}[1]_{q,1/q})
\end{aligned}$$

Note that

$$q^{12}[13]_{q,1/q} - q^{6}[7]_{q,1/q} = 1 + q^2 + q^4 \cdots + q^{24} - (1 + q^2 + q^4 + \cdots q^{12}) \equiv 0 \mod q^6$$

$$q^{12}[9]_{q,1/q} - q^{6}[3]_{q,1/q} = q^4 + q^6 \cdots + q^{20} - (q^4 + q^6 + \cdots q^{8}) \equiv 0 \mod q^6$$

$$q^{12}[5]_{q,1/q} + q^{12}[1]_{q,1/q} = q^8 + q^{10} + q^{12} + q^{16} + q^{18} + q^{12} \equiv 0 \mod q^6$$

Hence

$$q^{18}B_{3,4}(q,1/q) - q^{15}B_{3,3}(q,1/q) - R_{3,4}(q,1/q) \equiv 0 \mod q^{18}$$

Which proves (4.2).

4.1.2 Formal proof of the recurrence relation

Now we give the formal proof as follows:

Proof. Prove by induction.

(1) First, prove the relation holds for $k = 1$.

$$\begin{aligned} B_{3,1}(q,1/q) &= \tfrac{(1-q^{12})(1-q^{10})(1-q^{8})}{(1-q^{2})(1-q^{4})(1-q^{6})q^{9}} \\ &= q^9 + q^7 + 2q^5 + 3q^3 + 3q + 3q^{-1} + 3q^{-3} + 2q^{-5} + q^{-7} + q^{-9} \\ &= [10]_{q,1/q} + [6]_{q,1/q} + [4]_{q,1/q}. \end{aligned}$$

(2) Suppose that the relation holds for $k < j$. Now we prove that it also hold for $k = j$ $(j \geq 2)$.

Suppose $B_{3,j-1} = \sum_i [x_i]_{q,1/q}$. As a notation, we let

$$B'_{3,j}(q,1/q) = \sum_i [x_i + 3]_{q,1/q} + R_{3,j}(q,1/q)$$

Note that $B'_{3,j}(q,1/q)$ is a sum of sl_2-strings, and that we have

$$\begin{aligned} B_{3,k}(q,1/q) &= \tfrac{(1-(q^2)^{k+5})(1-(q^2)^{k+4})(1-(q^2)^{k+3})}{(1-q^2)(1-(q^2)^2)(1-(q^2)^3)q^{3k+6}} \\ &= \tfrac{(1-((1/q)^2)^{k+5})(1-((1/q)^2)^{k+4})(1-((1/q)^2)^{k+3})}{(1-(1/q)^2)(1-((1/q)^2)^2)(1-((1/q)^2)^3)(1/q)^{3k+6}} \end{aligned}$$

which gives that both $B_{3,j}(q,1/q)$ and $B'_{3,j}(q,1/q)$ has the same coefficients for q^m and $(1/q)^m$ for any $m \in \mathbb{N}^*$. Hence we only need to prove that $B_{3,j}(q,1/q) - B'_{3,j}(q,1/q)$ has coefficient 0 for any q^m, $m \in \mathbb{N}^*$, which is equivalent to

$$(q^{3j+6})B_{3,j}(q,1/q) \equiv q^{3j+6}B'_{3,j}(q,1/q) \mod q^{3j+6}.$$

Now we consider $B_{3,j-1}(q,1/q)$.

Case (a) $j \equiv 0, 1, 2 \mod 4$

Recall

$$q^{3j+3} B_{3,j-1}(q, 1/q) = \sum_i q^{3j+3} [x_i]_{q,1/q}.$$

and

$$B'_{3,j}(q, 1/q) = \sum_i [x_i + 3]_{q,1/q} + R_{3,j}(q, 1/q).$$

Compare

$$q^{3j+3} [x_i]_{q,1/q} = q^{3j+4-x_i} + q^{3j+6-x_i} + \cdots + q^{3j+x_i} + q^{3j+2+x_i}$$

and

$$q^{3j+6} [x_i + 3]_{q,1/q} = q^{3j+4-x_i} + q^{3j+6-x_i} + \cdots + q^{3j+6+x_i} + q^{3j+8+x_i},$$

We have

$$q^{3j+6} [x_i + 3]_{q,1/q} - q^{3j+3} [x_i]_{q,1/q} = q^{3j+4+x_i} + q^{3j+6+x_i} + \cdots + q^{3j+6+x_i} + q^{3j+8+x_i}.$$

Note that $j - 1 \equiv 1, 2, 3 \mod 4$, which gives that all $x_i \geq 3$, Hence $3j + 4 + x_i \geq 3j + 6$,

$$q^{3j+6} [x_i + 3]_{q,1/q} - q^{3j+3} [x_i]_{q,1/q} \equiv 0 \mod q^{3j+6}.$$

Hence we only need to prove that

$$q^{3j+6}B_{3,j}(q,1/q) \equiv q^{3j+3}B_{3,j-1}(q,1/q) + q^{3j+6}R_{3,j}(q,1/q) \mod q^{3j+6}. \tag{4.4}$$

Note that

$$q^{3j+6}B_{3,j}(q,1/q) = \frac{(1-q^{2j+10})(1-q^{2j+8})(1-q^{2j+6})}{(1-q^2)(1-q^4)(1-q^6)}$$

$$q^{3j+3}B_{3,j-1}(q,1/q) = \frac{(1-q^{2j+8})(1-q^{2j+6})(1-q^{2j+4})}{(1-q^2)(1-q^4)(1-q^6)}$$

Hence

$$q^{3j+6}B_{3,j}(q,1/q) - q^{3j+3}B_{3,j-1}(q,1/q) = q^{2j+4}\frac{(1-q^{2j+8})(1-q^{2j+6})}{(1-q^2)(1-q^4)}$$

From the previous calculation we know that

$$B_{2,j+1}(q,1/q) = \frac{(1-q^{2j+8})(1-q^{2j+6})}{(1-q^2)(1-q^4)q^{2j+4}} = [2j+5]_{q,1/q}+[2j+1]_{q,1/q}+\cdots+[1]_{q,1/q}(\text{or } [3]_{q,1/q})$$

Hence

$$q^{3j+6}B_{3,j}(q,1/q)-q^{3j+5}B_{3,j-1}(q,1/q) = q^{2j+4}(q^{2j+4}[2j+5]_{q,1/q}+\cdots+q^{2j+4}[1]_{q,1/q}(\text{or } [3]_{q,1/q}).$$

Case (a1) $j \equiv 0 \mod 4$

$$q^{3j+6}R_{3,j}(q,1/q) = q^{2j+4}(q^{j+2}[j+3]_{q,1/q} + \cdots + q^{j+2}[3]_{q,1/q})$$

We have

$$\begin{aligned}
&q^{3j+6}B_{3,j}(q,1/q) - q^{3j+3}B_{3,j}(q,1/q) - R_{3,j}(q,1/q)\\
&= q^{2j+4}(q^{2j+4}[2j+5]_{q,1/q} + q^{2j+4}[2j+1]_{q,1/q} + \cdots + q^{2j+4}[1]_{q,1/q}\\
&\qquad - q^{j+2}[j+3]_{q,1/q} - \cdots - q^{j+2}[3]_{q,1/q})\\
&= q^{2j+4}(q^{2j+4}[2j+5]_{q,1/q} - q^{j+2}[j+3]_{q,1/q}) + q^{2j+4}(q^{2j+4}[2j+1]_{q,1/q} - q^{j+2}[j-1]_{q,1/q})\\
&\qquad + \cdots + (q^{2j+4}[j+5]_{q,1/q} - q^{j+2}[3]_{q,1/q}) + q^{2j+4}(q^{2j+4}[j+1]_{q,1/q} + \cdots + q^{2j+4}[1]_{q,1/q})
\end{aligned}$$

Note that

$$q^{2j+4}[2j+5]_{q,1/q} - q^{j+2}[j+3]_{q,1/q} = 1+q^2+q^4\cdots+q^{4j+8} - (1+q^2+q^4+\cdots q^{2j+4}) \equiv 0 \mod q^{j+2}$$

$$q^{2j+4}[2j+1]_{q,1/q} - q^{j+2}[j-1]_{q,1/q} = q^4+q^6\cdots+q^{4j+4} - (q^4+q^6+\cdots q^{2j}) \equiv 0 \mod q^{j+2}$$

$$\vdots$$

$$q^{2j+4}[j+5]_{q,1/q} - q^{j+2}[3]_{q,1/q} = q^j+q^{j+2}+\cdots+q^{3j+8} - (q^j+q^{j+2}+q^{j+4}) \equiv 0 \mod q^{j+2}$$

Also note that

$$q^{2j+4}[x]_{q,1/q} \equiv 0 \mod q^{j+2}$$

for any $x \leq j+1$. Hence we can conclude that

$$q^{3j+6}B_{3,j}(q,1/q) - q^{3j+3}B_{3,j-1}(q,1/q) - R_{3,j}(q,1/q) \equiv 0 \mod q^{3j+6}$$

Which proves (4.4).

Similarly, we can prove the cases (a2) and (a3).

Case (a2) $j \equiv 1 \mod 4$

$$q^{3j+6}R_{3,j}(q,1/q) = q^{2j+4}(q^{j+2}[j+3]_{q,1/q} + \cdots + q^{j+2}[4]_{q,1/q})$$

We have

$$\begin{aligned}
&q^{3j+6}B_{3,j}(q,1/q) - q^{3j+3}B_{3,j}(q,1/q) - R_{3,j}(q,1/q)\\
&= q^{2j+4}(q^{2j+4}[2j+5]_{q,1/q} + q^{2j+4}[2j+1]_{q,1/q} + \cdots + q^{2j+4}[3]_{q,1/q}\\
&\qquad - q^{j+2}[j+3]_{q,1/q} - \cdots - q^{j+2}[4]_{q,1/q})\\
&= q^{2j+4}(q^{2j+4}[2j+5]_{q,1/q} - q^{j+2}[j+3]_{q,1/q}) + q^{2j+4}(q^{2j+4}[2j+1]_{q,1/q} - q^{j+2}[j-1]_{q,1/q})\\
&\qquad + \cdots + (q^{2j+4}[j+6]_{q,1/q} - q^{j+2}[4]_{q,1/q}) + q^{2j+4}(q^{2j+4}[j+2]_{q,1/q} + \cdots + q^{2j+4}[3]_{q,1/q})
\end{aligned}$$

Note that

$$q^{2j+4}[2j+5]_{q,1/q} - q^{j+2}[j+3]_{q,1/q} = 1+q^2+q^4\cdots+q^{4j+8} - (1+q^2+q^4+\cdots q^{2j+4}) \equiv 0 \mod q^{j+2}$$

$$q^{2j+4}[2j+1]_{q,1/q} - q^{j+2}[j-1]_{q,1/q} = q^4 + q^6 \cdots + q^{4j+4} - (q^4 + q^6 + \cdots q^{2j}) \equiv 0 \mod q^{j+2}$$

$$\vdots$$

$$q^{2j+4}[j+6]_{q,1/q} - q^{j+2}[4]_{q,1/q} = q^{j-1} + q^{j+1} + \cdots + q^{3j+9} - (q^{j-1} + q^{j+1} + q^{j+3} + q^{j+5}) \equiv 0 \mod q^{j+2}$$

Also note that

$$q^{2j+4}[x]_{q,1/q} \equiv 0 \mod q^{j+2}$$

for any $x \leq j+2$. Hence we can conclude that

$$q^{3j+6}B_{3,j}(q,1/q) - q^{3j+3}B_{3,j-1}(q,1/q) - R_{3,j}(q,1/q) \equiv 0 \mod q^{3j+6}$$

Which proves (4.4).

Case (a3) $j \equiv 2 \mod 4$

$$q^{3j+6}R_{3,j}(q,1/q) = q^{2j+4}(q^{j+2}[j+3]_{q,1/q} + \cdots + q^{j+2}[1]_{q,1/q})$$

We have

$$
\begin{aligned}
& q^{3j+6}B_{3,j}(q,1/q) - q^{3j+3}B_{3,j}(q,1/q) - R_{3,j}(q,1/q) \\
= {} & q^{2j+4}(q^{2j+4}[2j+5]_{q,1/q} + q^{2j+4}[2j+1]_{q,1/q} + \cdots + q^{2j+4}[1]_{q,1/q} \\
& \qquad - q^{j+2}[j+3]_{q,1/q} - \cdots - q^{j+2}[1]_{q,1/q}) \\
= {} & q^{2j+4}(q^{2j+4}[2j+5]_{q,1/q} - q^{j+2}[j+3]_{q,1/q}) + q^{2j+4}(q^{2j+4}[2j+1]_{q,1/q} - q^{j+2}[j-1]_{q,1/q}) \\
& \qquad + \cdots + (q^{2j+4}[j+3]_{q,1/q} - q^{j+2}[1]_{q,1/q}) + q^{2j+4}(q^{2j+4}[j-1]_{q,1/q} + \cdots + q^{2j+4}[1]_{q,1/q})
\end{aligned}
$$

Note that

$$q^{2j+4}[2j+5]_{q,1/q} - q^{j+2}[j+3]_{q,1/q} = 1+q^2+q^4\cdots+q^{4j+8} - (1+q^2+q^4+\cdots q^{2j+4}) \equiv 0 \mod q^{j+2}$$

$$q^{2j+4}[2j+1]_{q,1/q} - q^{j+2}[j-1]_{q,1/q} = q^4+q^6\cdots+q^{4j+4} - (q^4+q^6+\cdots q^{2j}) \equiv 0 \mod q^{j+2}$$

$$\vdots$$

$$q^{2j+4}[j+3]_{q,1/q} - q^{j+2}[1]_{q,1/q} = q^{j+2} + q^{j+4} + \cdots + q^{3j+6} - (q^{j+2}) \equiv 0 \mod q^{j+2}$$

Also note that

$$q^{2j+4}[x]_{q,1/q} \equiv 0 \mod q^{j+2}$$

for any $x \leq j-1$. Hence we can conclude that

$$q^{3j+6}B_{3,j}(q,1/q) - q^{3j+3}B_{3,j-1}(q,1/q) - R_{3,j}(q,1/q) \equiv 0 \mod q^{3j+6}$$

Which proves (4.4).

Case (b) $j \equiv 3 \mod 4$

This case is slightly different from (a).

Compare

$$q^{3j+3}[x_i]_{q,1/q} = q^{3j+4-x_i} + q^{3j+6-x_i} + \cdots + q^{3j+x_i} + q^{3j+2+x_i}$$

and

$$q^{3j+6}[x_i+3]_{q,1/q} = q^{3j+4-x_i} + q^{3j+6-x_i} + \cdots + q^{3j+6+x_i} + q^{3j+8+x_i},$$

we have

$$q^{3j+6}[x_i+3]_{q,1/q} - q^{3j+3}[x_i]_{q,1/q} = q^{3j+4+x_i} + q^{3j+6+x_i} + \cdots + q^{3j+6+x_i} + q^{3j+8+x_i}.$$

Note that $j-1 \equiv 2 \mod 4$, which gives that $x_0 = 1$ and all other $x_i \geq 3$.

For those $x_i \geq 3$, $3j+4+x_i \geq 3j+6$,

$$q^{3j+6}[x_i+3]_{q,1/q} - q^{3j+3}[x_i]_{q,1/q} \equiv 0 \mod q^{3j+6}.$$

for $x_0 = 1$,

$$q^{3j+6}[4]_{q,1/q} - q^{3j+3}[1]_{q,1/q} \equiv q^{3j+5} \mod q^{3j+6}.$$

Hence we need to prove that

$$q^{3j+6}B_{3,j}(q,1/q) \equiv q^{3j+3}B_{3,j-1}(q,1/q) + q^{3j+6}(R_{3,j}(q,1/q)+1/q) \mod q^{3j+6}. \quad (4.5)$$

Note that $[2]_{q,1/q} = q + 1/q$, By letting $R'_{3,j}(q,1/q) = R_{3,j}(q,1/q) + [2]_{q,1/q}$, (4.5) is equivalent to

$$q^{3j+6}B_{3,j}(q,1/q) \equiv q^{3j+3}B_{3,j-1}(q,1/q) + q^{3j+6}(R'_{3,j}(q,1/q)) \mod q^{3j+6}. \quad (4.6)$$

Now the proof is similar to case (a).

$$q^{3j+6}B_{3,j}(q,1/q) - q^{3j+5}B_{3,j-1}(q,1/q) = q^{2j+4}(q^{2j+4}[2j+5]_{q,1/q} + \cdots + q^{2j+4}[3]_{q,1/q}.$$

$$q^{3j+6}R'_{3,j}(q,1/q) = q^{2j+4}(q^{j+2}[j+3]_{q,1/q} + \cdots + q^{j+2}[6]_{q,1/q} + q^{j+2}[2]_{q,1/q})$$

We have

$$\begin{aligned}
&q^{3j+6}B_{3,j}(q,1/q) - q^{3j+3}B_{3,j}(q,1/q) - R'_{3,j}(q,1/q) \\
&= q^{2j+4}(q^{2j+4}[2j+5]_{q,1/q} + q^{2j+4}[2j+1]_{q,1/q} + \cdots + q^{2j+4}[3]_{q,1/q} \\
&\qquad - q^{j+2}[j+3]_{q,1/q} - \cdots - q^{j+2}[2]_{q,1/q}) \\
&= q^{2j+4}(q^{2j+4}[2j+5]_{q,1/q} - q^{j+2}[j+3]_{q,1/q}) + q^{2j+4}(q^{2j+4}[2j+1]_{q,1/q} - q^{j+2}[j-1]_{q,1/q}) \\
&\qquad + \cdots + (q^{2j+4}[j+4]_{q,1/q} - q^{j+2}[2]_{q,1/q}) + q^{2j+4}(q^{2j+4}[j]_{q,1/q} + \cdots + q^{2j+4}[3]_{q,1/q})
\end{aligned}$$

Note that

$$q^{2j+4}[2j+5]_{q,1/q} - q^{j+2}[j+3]_{q,1/q} = 1+q^2+q^4\cdots+q^{4j+8} - (1+q^2+q^4+\cdots q^{2j+4}) \equiv 0 \mod q^{j+2}$$

$$q^{2j+4}[2j+1]_{q,1/q} - q^{j+2}[j-1]_{q,1/q} = q^4+q^6\cdots+q^{4j+4} - (q^4+q^6+\cdots q^{2j}) \equiv 0 \mod q^{j+2}$$

$$\vdots$$

$$q^{2j+4}[j+4]_{q,1/q} - q^{j+2}[2]_{q,1/q} = q^{j+1}+q^{j+3}+\cdots+q^{3j+7} - (q^{j+1}+q^{j+3}) \equiv 0 \mod q^{j+2}$$

Also note that

$$q^{2j+4}[x]_{q,1/q} \equiv 0 \mod q^j$$

for any $x \leq j+2$. Hence we can conclude that

$$q^{3j+6}B_{3,j}(q,1/q) - q^{3j+3}B_{3,j-1}(q,1/q) - R'_{3,j}(q,1/q) \equiv 0 \mod q^{3j+6}$$

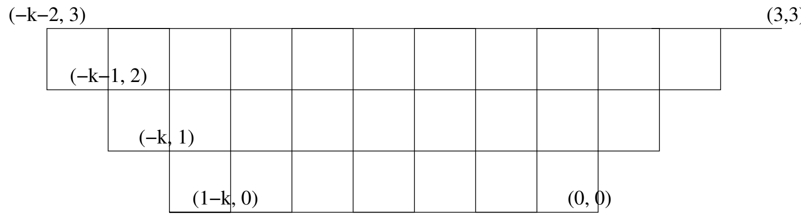
Which proves (4.6).

And now we've completed the proof. □

4.2 Area statistic and $B_{3,k}(1,t)$

The shifted Dyck path can start from $(1-k,0)$, $(-k,1)$, $(-k-1,2)$ or $(-k-2,3)$.

- case 1: starting from $(-k-2,3)$. Apparently, the only path here is taking every



step East.

$$\sum_{\substack{\pi\in\mathcal{E}_{3,k},\\ \text{start point }=(-k-2,3)}} t^{\text{area}} = t^{3k+6}.$$

- case 2: starting from $(-k,1)$. There are $k+1$ choices to take the step North.

$$\sum_{\substack{\pi\in\mathcal{E}_{3,k},\\ \text{start point }=(-k-1,2)}} t^{\text{area}} = t^{2k+2}(1+t+t^2+\cdots+t^{k+3}) = t^{2k+2}[k+4]_t.$$

- case 3 starting from $(k,1)$. There are $k+1$ choices to take the first step North, and then the situation reduces to one in case 2.

$$\sum_{\substack{\pi\in\mathcal{E}_{3,k},\\ \text{start point }=(-k,1)}} t^{\text{area}} = t^{2k+1}[k+3]_t + t^{2k}[k+2]_t + \cdots + t^k[2]_t.$$

- case 4 starting from $(1-k,0)$. There are k choices to take the first step North, and then the situation reduces to one in case 3.

$$\begin{aligned}\sum_{\substack{\pi\in\mathcal{E}_{3,k},\\ \text{start point }=(1-k,0)}} t^{\text{area}} = \; & t^{k-1}(t^{k}[k+2]_t + t^{k-1}[k+1]_t + \cdots + t^{0}[2]_t)\\ & + t^{k-2}(t^{k-1}[k+1]_t + t^{k-2}[k]_t + \cdots + t^{0}[2]_t)\\ & + \cdots + t^{0}(t^{1}[3]_t + t^{0}[2]_t).\\ = \; & t^{2k-1}[k+2]_t + t^{2k-3}[2]_t[k+1]_t + t^{2k-5}[3]_t[k]_t\\ & + \cdots + t^{1}[k]_t[3]_t + t^{0}[k]_t[2]_t.\end{aligned}$$

Hence by adding them up together we have

$$\begin{aligned}B_{3,k}(1,t) = \; & \textstyle\sum_{\pi\in\mathcal{E}_{3,k}} t^{\text{area}}\\ = \; & t^{2k+2}[k+5]_t + t^{2k+1}[k+3]_t + t^{2k-1}[2]_t[k+2]_t + t^{2k-3}[3]_t[k+1]_t\\ & + t^{2k-5}[4]_t[k]_t + \cdots + t^{1}[k+1]_t[3]_t + t^{0}[k+1]_t[2]_t\\ = \; & t^{2k+2}[k+5]_t + (\textstyle\sum_{i=1}^{k+1} t^{2k+3-2i}[i]_t[k+4-i]_t) + t^{0}[k+1]_t[2]_t.\end{aligned}$$

4.3 Recurrence relation of $B_{3,k}(1,t)$

We now simplify this formula and write it as a sum of sl_2-strings. The main result of this part is the recurrence relation stated and proved as follows:

Theorem 4.3. *$B_{3,k}(1,t)$ follows the following recursive form:*

(1) $B_{3,1}(1,t)$

$$B_{3,1}(1,t) = [10]_t + t[6]_t + t[4]_t.$$

(2) Suppose $B_{3,k}(1,t) = \sum_i t^{m_i}[x_i]_t$, *then*

$$B_{3,k+1}(1,t) = \sum_i t^{m_j}[x_i+3]_t + R_{3,k+1}(1,t)$$

where $R_{3,k}$ *is additional terms for* $k \geq 2$ *as follows:*

$$R_{3,k}(1,t) = \begin{cases} t^k[k+3]_t + t^{k+1}[k-1]_t & +t^{k+3}[k-5]_t \\ & + \cdots + t^{\frac{3k-4}{2}}[5]_t + t^{\frac{3k}{2}}[1]_t \qquad k \equiv 2 \mod 4 \\ t^k\,[k+3]_t + t^{k+1}[k-1]_t & +t^{k+3}[k-5]_t \\ & + \cdots + t^{\frac{3k-9}{2}}[10]_t + t^{\frac{3k-5}{2}}[6]_t \quad k \equiv 3 \mod 4 \\ t^k\,[k+3]_t + t^{k+1}[k-1]_t & +t^{k+3}[k-5]_t \\ & + \cdots + t^{\frac{3k-6}{2}}[7]_t + t^{\frac{3k-2}{2}}[3]_t \qquad k \equiv 0 \mod 4 \\ t^k\,[k+3]_t + t^{k+1}[k-1]_t & +t^{k+3}[k-5]_t \\ & + \cdots + t^{\frac{3k-7}{2}}[8]_t + t^{\frac{3k-3}{2}}[4]_t \qquad k \equiv 1 \mod 4 \end{cases}$$

We will prove the theorem by induction. Again, we will first give an example for $k = 6$ to help understand the frame of the proof, and then we will state the mathematic proof strictly.

4.3.1 Example: $B_{3,6}(1,t)$

Example 4.4. As a notation, we let $B'_{3,k}(1,t)$ be the right hand side of the recursive form. Then

$$B'_{3,1}(1,t) = B_{3,1}(1,t) = [10]_t + t[6]_t + t[4]_t. \tag{4.7}$$

$$B'_{3,2}(1,t) = [13]_t + t[9]_t + t[7]_t + t^2[5]_t + t^3[1]_t. \tag{4.8}$$

$$B'_{3,3}(1,t) = [16]_t + t[12]_t + t[10]_t + t^2[8]_t + t^3[4]_t + t^3[6]_t. \tag{4.9}$$

$$B'_{3,4}(1,t) = [19]_t + t[15]_t + t[13]_t + t^2[11]_t + t^3[9]_t + t^3[7]_t + t^4[7]_t + t^5[3]. \tag{4.10}$$

$$B'_{3,5}(1,t) = [22]_t + t[18]_t + t[16]_t + t^2[14]_t + t^3[12]_t + t^3[10]_t + t^4[10]_t \tag{4.11}$$

$$+ t^5[6]_t + t^5[8]_t + t^6[4]_t. \tag{4.12}$$

$$B'_{3,6}(1,t) = [25]_t + t[21]_t + t[19]_t + t^2[17]_t + t^3[15]_t + t^3[13]_t + t^4[13]_t \tag{4.13}$$

$$+ t^5[9]_t + t^5[11]_t + t^6[7]_t + t^6[9]_t + t^7[5]_t + t^9[1]_t. \tag{4.14}$$

Now we prove $B_{3,6}(1,t) = B'_{3,6}(1,t)$ by induction. Suppose that $B_{3,k}(1,t) = B'_{3,k}(1,t)$ for $k = 1, 2, 3, 4, 5$. Now

$$\begin{aligned} B'_{3,6}(1,t) - B'_{3,5}(1,t) = \quad & (1+t+t^2)(t^{22} + t \cdot t^{18} + t \cdot t^{16} + t^2 \cdot t^{14} + t^3 \cdot t^{12} + t^3 \cdot t^{10} \\ & + t^4 \cdot t^{10} + t^5 \cdot t^6 + t^5 \cdot t^8 + t^4 \cdot t^4) + t^6[9]_t + t^7[5]_t + t^9[1]_t. \end{aligned}$$

$$\begin{aligned} B'_{3,5}(1,t) - B'_{3,4}(1,t) = & \quad (1+t+t^2)(t^{19} + t\cdot t^{15} + t\cdot t^{13} + t^2\cdot t^{11} + t^3\cdot t^9 + t^3\cdot t^7 \\ & \quad + t^4\cdot t^7 + t^5\cdot t^3) + t^5[8]_t + t^6[4]_t. \\ = & \quad (1+t+t^2)(t^{19} + t\cdot t^{15} + t\cdot t^{13} + t^2\cdot t^{11} + t^3\cdot t^9 + t^3\cdot t^7 \\ & \quad + t^4\cdot t^7 + t^5\cdot t^3 + t^5\cdot t^5 + t^4\cdot t^1) + t^5[5]_t + t^6[1]_t. \end{aligned}$$

Hence

$$\begin{aligned} & B'_{3,6}(1,t) - B'_{3,5}(1,t) - t^3(B'_{3,5}(1,t) - B'_{3,4}(1,t)) \\ = \quad & t^6[9]_t + t^7[5]_t + t^9[1]_t - t^3(t^5[5]_t + t^6[1]_t) \\ = \quad & (t^6[9]_t - t^8[5]_t) + (t^7[5]_t - t^6[1]_t) + t^9[1]_t \\ = \quad & t^6 + t^7 + t^{13} + t^14 + t^7 + t^8 + t^{10} + t^{11} + t^9 \\ = \quad & t^6[9]_t + t^7 - t^{12}. \end{aligned} \tag{4.15}$$

On the other hand, we have

$$\begin{aligned} B_{3,4}(1,t) =& t^{10}[9]_t + t^9[7]_t + t^7[2]_t[6]_t + t^5[3]_t[5]_t + t^3[4]_t[4]_t + t[5]_t[3]_t \\ & + [5]_t[2]_t. \\ B_{3,5}(1,t) =& t^{12}[10]_t + t^{11}[8]_t + t^9[2]_t[7]_t + t^7[3]_t[6]_t + t^5[4]_t[5]_t + t^3[5]_t[4]_t \\ & + t[6]_t[3]_t + [6]_t[2]_t. \\ B_{3,6}(1,t) =& t^{14}[11]_t + t^{13}[9]_t + t^{11}[2]_t[8]_t + t^9[3]_t[7]_t + t^7[4]_t[6]_t + t^5[5]_t[5]_t \\ & + t^3[6]_t[4]_t + t[7]_t[3]_t + [7]_t[2]_t. \end{aligned}$$

Then

$$
\begin{aligned}
B_{3,6}(1,t)-B_{3,5}(1,t)= \;& t^{12}(t^{12}+t^{11}+t^{10}-t-1)+t^{11}(t^{10}+t^{9}+t^{8}-t-1)\\
&+t^{9}[2]_t(t^{9}+t^{8}+t^{7}-t-1)+\cdots+t^{3}[5]_t(t^{6}+t^{5}+t^{4}-t-1)\\
&+t[6]_t(t^{5}+t^{4}+t^{3}-t-1)+t[7]_t[3]_t+t^{6}(1+t).
\end{aligned}
$$

$$
\begin{aligned}
B_{3,5}(1,t)-B_{3,4}(1,t)= \;& t^{10}(t^{11}+t^{10}+t^{9}-t-1)+t^{9}(t^{9}+t^{8}+t^{7}-t-1)\\
&+t^{7}[2]_t(t^{8}+t^{6}+t^{5}-t-1)+\cdots+t[5]_t(t^{5}+t^{4}+t^{3}-t-1)\\
&+t[6]_t[3]_t+t^{5}(1+t).
\end{aligned}
$$

Hence

$$
\begin{aligned}
& B_{3,6}(1,t)-B_{3,5}(1,t)-t^{3}(B_{3,5}(1,t)-B_{3,4}(1,t))\\
= \;& (t^{2}-1)(t^{12}+t^{11}+t^{9}[2]_t+\cdots+t^{3}[5]_t)+t[6]_t(t^{5}+t^{4}+t^{3}-t-1)\\
& -(t^{9}+t^{8}-t^{3}-t^{2}-t)[3]_t-(t^{8}-t^{6})(1+t)\\
= \;& (1+t)(t^{11}(t^{2}-1)+t^{9}(t^{2}-1)+t^{7}(t^{3}-1)+t^{5}(t^{4}-1)+t^{3}(t^{5}-1))\\
& +(t^{4}+2t^{5}+3t^{6}+3t^{7}+3t^{8}+3t^{9}+2t^{10}+t^{11}-t-2t^{2}-2t^{3}-2t^{4}-2t^{5}-2t^{6}-t^{7})\\
& +(t+2t^{2}+3t^{3}+2t^{4}+t^{5}-t^{8}-2t^{9}-2t^{10}-t^{11})+(t^{6}+t^{7}-t^{8}-t^{9})\\
= \;& (t^{14}+t^{13}+t^{11}+t^{10}+t^{9}-t^{7}-t^{6}-t^{5}-t^{4}-t^{3})+(t^{4}+t^{5}+2t^{6}+3t^{7}+t^{8}+t^{3})\\
= \;& t^{14}+t^{13}+t^{11}+t^{10}+t^{9}+t^{8}+2t^{7}+t^{6}\\
= \;& t^{6}[9]_t+t^{7}-t^{12}.
\end{aligned}
$$

Compare with (4.15) we have

$$\begin{aligned} & B_{3,6}(1,t) - B_{3,5}(1,t) - t^3(B_{3,5}(1,t) - B_{3,4}(1,t)) \\ = \; & B'_{3,6}(1,t) - B'_{3,5}(1,t) - t^3(B'_{3,5}(1,t) - B'_{3,4}(1,t)). \end{aligned}$$

Note that $B_{3,k}(1,t) = B'_{3,k}(1,t)$ for $k = 4$ and 5, we can conclude that

$$B_{3,6}(1,t) = B'_{3,6}(1,t).$$

4.3.2 Formal proof of the recurrence relation

Now we give the formal proof as follows:

Proof. Prove by induction.

(1) First, prove the relation holds for $k = 1$.

$$\begin{aligned} B_{3,1}(1,t) &= t^4[6]_t + t^3[4]_t + t[2]_t[3]_t + [2]_t[2]_t \\ &= [10]_t + t[6]_t + t[4]_t \end{aligned}$$

(2) Suppose that the relation holds for $k \leq j$. Now we prove that it also hold for $k = j+1$ ($j \geq 1$).

Suppose $B_{3,k}(1,t) = \sum_{i \in S_k} t^{m_{k,i}}[x_{k,i}]_t$. As a notation, we let

$$B'_{3,k+1}(1,t) = \sum_{i \in S_k} t^{m_{k,i}}[x_{k,i}+3]_t + R_{3,k+1}(1,t)$$

Now we have $B_{3,k}(1,t) = B'_{3,k}(1,t)$ for all $k = 1, 2, \cdots, j$. Our goal is to prove that $B_{3,j+1}(1,t) = B'_{3,j+1}(1,t)$.

First,

$$B'_{3,j+1}(1,t) - B'_{3,j}(1,t) = (1+t+t^2)(\sum_{i\in S_j} t^{m_{j,i}}[x_{j,i}]_t) + R_{3,j+1}(1,t).$$

$$\begin{aligned} B'_{3,j}(1,t) - B'_{3,j-1}(1,t) &= (1+t+t^2)(\sum_{i\in S_{j-1}} t^{m_{j-1,i}}[x_{j-1,i}]_t) + R_{3,j}(1,t) \\ &= (1+t+t^2)(\sum_{j\in S_j} t^{m_{j,i}}[x_{j,i}-3]_t) + \tilde{R}_{3,j}(1,t), \end{aligned}$$

where

$$\tilde{R}_{3,k}(1,t) = \begin{cases} t^k[k]_t + t^{k+1}[k-4]_t & +t^{k+3}[k-8]_t \\ & + \cdots + t^{\frac{3k-4}{2}}[2]_t + t^{\frac{3k}{2}}[-2]_t \quad k \equiv 2 \mod 4 \\ t^k\,[k]_t + t^{k+1}[k-4]_t & +t^{k+3}[k-8]_t \\ & + \cdots + t^{\frac{3k-9}{2}}[7]_t + t^{\frac{3k-5}{2}}[3]_t \quad k \equiv 3 \mod 4 \\ t^k\,[k]_t + t^{k+1}[k-4]_t & +t^{k+3}[k-8]_t \\ & + \cdots + t^{\frac{3k-6}{2}}[4]_t \quad k \equiv 0 \mod 4 \\ t^k\,[k]_t + t^{k+1}[k-4]_t & +t^{k+3}[k-8]_t \\ & + \cdots + t^{\frac{3k-7}{2}}[5]_t + t^{\frac{3k-3}{2}}[1]_t \quad k \equiv 1 \mod 4 \end{cases}$$

Hence

$$B'_{3,j+1}(1,t) - B'_{3,j}(1,t) - t^3(B'_{3,j}(1,t) - B'_{3,j-1}(1,t)) = R_{3,j+1}(1,t) - t^3(\tilde{R}_{3,j}(1,t)) \quad (4.16)$$

Case (a): When $j \equiv 2 \mod 4$, (4.16) will be

$$\begin{aligned}
& (t^{j+1}[j+4]_t - t^3 \cdot t^j[j]_t) + (t^{j+2}[j]_t - t^3 \cdot t^{j+1}[j-4]_t) + (t^{j+4}[j-4]_t - t^3 \cdot t^{j+3}[j-8]_t) \\
& + \cdots + (t^{\frac{3j-2}{2}}[6]_t - t^3 \cdot t^{\frac{3j-4}{2}}[2]_t) - t^3 \cdot t^{\frac{3j}{2}}[-2]_t \\
= \; & (t^{2j+4} + t^{2j+3} + t^{j+2} + t^{j+1}) + (t^{2j+1} + t^{2j} + t^{j+3} + t^{j+2}) + (t^{2j-1} + t^{2j-2} + t^{j+5} + t^{j+4}) \\
& + \cdots + (t^{\frac{3j}{2}+4} + t^{\frac{3j}{2}+3} + t^{\frac{3j}{2}} + t^{\frac{3j}{2}-1}) + (t^{\frac{3j}{2}+2} + t^{\frac{3j}{2}+1}) \\
= \; & t^{j+1}[j+4]_t - t^{2j+2} + t^{j+2}.
\end{aligned}$$

Similarly, we have

Case (b): When $j \equiv 3 \mod 4$, (4.16) will be

$$\begin{aligned}
& (t^{j+1}[j+4]_t - t^3 \cdot t^j[j]_t) + (t^{j+2}[j]_t - t^3 \cdot t^{j+1}[j-4]_t) + (t^{j+4}[j-4]_t - t^3 \cdot t^{j+3}[j-8]_t) \\
& + \cdots + (t^{\frac{3j-3}{2}}[7]_t - t^3 \cdot t^{\frac{3j-5}{2}}[3]_t) + t^{\frac{3j+1}{2}}[3]_t \\
= \; & (t^{2j+4} + t^{2j+3} + t^{j+2} + t^{j+1}) + (t^{2j+1} + t^{2j} + t^{j+3} + t^{j+2}) + (t^{2j-1} + t^{2j-2} + t^{j+5} + t^{j+4}) \\
& + \cdots + (t^{\frac{3j+9}{2}} + t^{\frac{3j+7}{2}} + t^{\frac{3j-1}{2}} + t^{\frac{3j-3}{2}}) + (t^{\frac{3j+5}{2}} + t^{\frac{3j+3}{2}} + t^{\frac{3j+1}{2}}) \\
= \; & t^{j+1}[j+4]_t - t^{2j+2} + t^{j+2}.
\end{aligned}$$

Case (c): When $j \equiv 0 \mod 4$, (4.16) will be

$$\begin{aligned}
& (t^{j+1}[j+4]_t - t^3 \cdot t^j[j]_t) + (t^{j+2}[j]_t - t^3 \cdot t^{j+1}[j-4]_t) + (t^{j+4}[j-4]_t - t^3 \cdot t^{j+3}[j-8]_t) \\
& + \cdots + (t^{\frac{3j-4}{2}}[8]_t - t^3 \cdot t^{\frac{3j-6}{2}}[4]_t) + t^{\frac{3j}{2}}[4]_t \\
= \; & (t^{2j+4} + t^{2j+3} + t^{j+2} + t^{j+1}) + (t^{2j+1} + t^{2j} + t^{j+3} + t^{j+2}) + (t^{2j-1} + t^{2j-2} + t^{j+5} + t^{j+4}) \\
& + \cdots + (t^{\frac{3j}{2}+5} + t^{\frac{3j}{2}+4} + t^{\frac{3j}{2}-1} + t^{\frac{3j}{2}-2}) + (t^{\frac{3j}{2}+3} + t^{\frac{3j}{2}+2} + t^{\frac{3j}{2}+1} + t^{\frac{3j}{2}}) \\
= \; & t^{j+1}[j+4]_t - t^{2j+2} + t^{j+2}.
\end{aligned}$$

Case (d): When $j \equiv 1 \mod 4$, (4.16) will be

$$\begin{aligned}
& (t^{j+1}[j+4]_t - t^3 \cdot t^j[j]_t) + (t^{j+2}[j]_t - t^3 \cdot t^{j+1}[j-4]_t) + (t^{j+4}[j-4]_t - t^3 \cdot t^{j+3}[j-8]_t) \\
& + \cdots + (t^{\frac{3j-1}{2}}[5]_t - t^3 \cdot t^{\frac{3j-3}{2}}[1]_t) + t^{\frac{3j+3}{2}}[1]_t \\
= \; & (t^{2j+4} + t^{2j+3} + t^{j+2} + t^{j+1}) + (t^{2j+1} + t^{2j} + t^{j+3} + t^{j+2}) + (t^{2j-1} + t^{2j-2} + t^{j+5} + t^{j+4}) \\
& + \cdots + (t^{\frac{3j+7}{2}} + t^{\frac{3j+5}{2}} + t^{\frac{3j+1}{2}} + t^{\frac{3j-1}{2}}) + (t^{\frac{3j+3}{2}}) \\
= \; & t^{j+1}[j+4]_t - t^{2j+2} + t^{j+2}.
\end{aligned}$$

Hence to sum up, we have

$$B'_{3,j+1}(1,t) - B'_{3,j}(1,t) - t^3(B'_{3,j}(1,t) - B'_{3,j-1}(1,t)) = t^{j+1}[j+4]_t - t^{2j+2} + t^{j+2}. \quad (4.17)$$

Now on the other hand, we have

$$\begin{aligned}
B_{3,j-1}(1,t) = \;& t^{2j}[j+4]_t + t^{2j-1}[j+2]_t + t^{2j-3}[2]_t[j+1]_t + t^{2j-5}[3]_t[j]_t \\
& + t^{2j-7}[4]_t[j-1]_t + \cdots + t^1[j]_t[3]_t + t^0[j]_t[2]_t \\
B_{3,j}(1,t) = \;& t^{2j+2}[j+5]_t + t^{2j+1}[j+3]_t + t^{2j-1}[2]_t[j+2]_t + t^{2j-3}[3]_t[j+1]_t \\
& + t^{2j-5}[4]_t[j]_t + \cdots + t^1[j+1]_t[3]_t + t^0[j+1]_t[2]_t \\
B_{3,j+1}(1,t) = \;& t^{2j+4}[j+6]_t + t^{2j+3}[j+4]_t + t^{2j+1}[2]_t[j+3]_t + t^{2j-1}[3]_t[j+2]_t \\
& + t^{2j-3}[4]_t[j+1]_t + \cdots + t^1[j+2]_t[3]_t + t^0[j+2]_t[2]_t
\end{aligned}$$

Then

$$\begin{aligned}
& B_{3,j+1}(1,t) - B_{3,j}(1,t) \\
= \;& t^{2j+2}(t^{j+7} + t^{j+6} + t^{j+5} - t - 1) + t^{2j+1}(t^{j+5} + t^{j+4} + t^{j+3} - t - 1) \\
& + t^{2j-1}[2]_t(t^{j+4} + t^{j+3} + t^{j+2} - t - 1) + \cdots + t^3[j]_t(t^6 + t^5 + t^4 - t - 1) \\
& + t[j+1]_t(t^5 + t^4 + t^3 - t - 1) + t[j+2]_t[3]_t + t^{j+1}(1+t).
\end{aligned}$$

$$\begin{aligned}
& B_{3,j}(1,t) - B_{3,j-1}(1,t) \\
= \;& t^{2j}(t^{j+6} + t^{j+5} + t^{j+4} - t - 1) + t^{2j-1}(t^{j+4} + t^{j+3} + t^{j+2} - t - 1) \\
& + t^{2j-3}[2]_t(t^{j+3} + t^{j+2} + t^{j+1} - t - 1) + \cdots \\
& + t[j]_t(t^5 + t^4 + t^3 - t - 1) + t[j+1]_t[3]_t + t^j(1+t).
\end{aligned}$$

Hence

$$\begin{aligned}
& B_{3,j+1}(1,t) - B_{3,j}(1,t) - t^3(B_{3,j}(1,t) - B_{3,j-1}(1,t)) \\
= \;& (t^2-1)(t^{2j+2} + t^{2j+1} + t^{2j-1}[2]_t + \cdots + t^3[j]_t) + t[j+1]_t(t^5+t^4+t^3-t-1) \\
& -(t^{j+4}+t^{j+3}-t^3-t^2-t)[3]_t - (t^{j+3}-t^{j+1})(1+t) \\
= \;& (1+t)(t^{2j+1}(t^2-1) + t^{2j-1}(t^2-1) + t^{2j-3}(t^3-1) + \cdots + t^3(t^j-1)) \\
& -(1+t)(t[j+1]_t) + [3]_t(t^4[j+1]_t - t^{j+4} - t^{j+3} + t[3]_t) - t^{j+4} - t^{j+3} + t^{j+2} + t^{j+1} \\
= \;& (t^{j+4}[j+1]_t - t^{2j+2} - t^3[j]_t - (1+t)(t[j+1]_t) + (1+t+t^2)(t[j+2]_t) \\
& -t^{j+4} - t^{j+3} + t^{j+2} + t^{j+1} \\
= \;& (t^{j+4}[j+1]_t - t^{2j+2} - t^3[j]_t + (1+t)(t^{j+2}) + (t^3)[j+2]_t - t^{j+4} - t^{j+3} + t^{j+2} + t^{j+1} \\
= \;& t^{j+4}[j+1]_t - t^{2j+2} + t^{j+4} + t^{j+3} - t^{j+4} + 2t^{j+2} + t^{j+1} \\
= \;& t^{j+1}[j+4]_t - t^{2j+2} + t^{j+2}.
\end{aligned}$$

Compare with (4.17) we have

$$\begin{aligned}
& B_{3,j+1}(1,t) - B_{3,j}(1,t) - t^3(B_{3,j}(1,t) - B_{3,j-1}(1,t)) \\
= \;& B'_{3,j+1}(1,t) - B'_{3,j}(1,t) - t^3(B'_{3,j}(1,t) - B'_{3,j-1}(1,t))
\end{aligned}$$

Note that $B_{3,j}(1,t) = B'_{3,j}(1,t)$ and $B_{3,j-1}(1,t) = B'_{3,j-1}(1,t)$, we can conclude that

$$B_{3,j+1}(1,t) = B'_{3,j+1}(1,t).$$

And now we've completed the proof. $\square$

4.4 A candidate description of $B_{3,k}(q,t)$ as a positive linear combination of sl_2-strings

By the conjectures above, we can give a candidate description of $B_{3,k}(q,t)$ as a sum of sl_2-strings as follows:

Theorem 4.5. *$B_{3,k}(q,t)$ follows the following recursive form:*

(1) $B_{3,1}(q,t)$

$$B_{3,1}(q,t) = [10]_{q,t} + (qt)[6]_{q,t} + (qt)[4]_{q,t}.$$

(2) Suppose $B_{3,k}(q,t) = \sum_i (qt)^{m_i}[x_i]_{q,t}$, then

$$B_{3,k+1}(q,t) = \sum_i (qt)^{m_i}[x_i+3]_{q,t} + R_{3,k+1}(q,t)$$

where $R_{3,k}$ is additional terms for $k \geq 2$ as follows:

$$R_{3,k}(q,t) = \begin{cases} (qt)^k[k+3]_{q,t} + (qt)^{k+1}[k-1]_{q,t} & +(qt)^{k+3}[k-5]_{q,t} \\ & +\cdots+(qt)^{\frac{3k}{2}}[1]_{q,t} \quad k \equiv 2 \mod 4 \\ (qt)^k[k+3]_{q,t} + (qt)^{k+1}[k-1]_{q,t} & +(qt)^{k+3}[k-5]_{q,t} \\ & +\cdots+(qt)^{\frac{3k-5}{2}}[6]_{q,t} \quad k \equiv 3 \mod 4 \\ (qt)^k[k+3]_{q,t} + (qt)^{k+1}[k-1]_{q,t} & +(qt)^{k+3}[k-5]_{q,t} \\ & +\cdots+(qt)^{\frac{3k-2}{2}}[3]_{q,t} \quad k \equiv 0 \mod 4 \\ (qt)^k[k+3]_{q,t} + (qt)^{k+1}[k-1]_{q,t} & +(qt)^{k+3}[k-5]_{q,t} \\ & +\cdots+(qt)^{\frac{3k-3}{2}}[4]_{q,t} \quad k \equiv 1 \mod 4 \end{cases}$$

5 Writing $B_{n,k}(q, 1/q)$ as a sum of sl_2-strings for all n, k

In this section we will derive a recurrence relation for writing $B_{n,k}(q, 1/q)$ as a sum of sl_2-strings for all n, k.

5.1 Recurrence relation

Recall the given assumption 2.1

$$q^{n(n+k-1)} B_{n,k}(q, 1/q) = \begin{bmatrix} 2n+k-1 \\ n \end{bmatrix}_{q^2}. \tag{5.1}$$

Note that by [1] Theorem 3.4, we have

$$\begin{bmatrix} n+1+m \\ n+1 \end{bmatrix}_q = \sum_{j=0}^{m} q^j \begin{bmatrix} n+j \\ n \end{bmatrix}_q, \qquad \text{for } m, n \geq 0. \tag{5.2}$$

Now we define $F_{n,m}(p) = \left[\begin{smallmatrix} m+n \\ n \end{smallmatrix}\right]_p p^{-nm/2}$. (under this notation we have $B_{n,k}(q, 1/q) = F_{n,n+k-1}(q^2)$.) Using (5.2), we have

$$\begin{aligned} F_{n,m}(q^2) &= \left[\begin{smallmatrix} m+n \\ n \end{smallmatrix}\right]_{q^2} q^{-nm} \\ &= \textstyle\sum_{j=0}^{m} q^{2j} \left[\begin{smallmatrix} j+n-1 \\ n-1 \end{smallmatrix}\right]_{q^2} q^{-nm} \\ &= \textstyle\sum_{j=0}^{m} \left[\begin{smallmatrix} j+n-1 \\ n-1 \end{smallmatrix}\right]_{q^2} q^{-(n-1)j} \cdot q^{-nm+(n+1)j} \\ &= \textstyle\sum_{j=0}^{m} q^{-nm+(n+1)j} F_{n-1,j}(q^2). \end{aligned}$$

Then by

$$F_{n,m-1}(q^2) = \sum_{j=0}^{m-1} q^{-n(m-1)+(n+1)j} F_{n-1,j}(q^2)$$

We have the following recurrence relation:

Theorem 5.1.

$$F_{0,m}(q^2) = F_{n,0}(q^2) = 1,$$

$$F_{n,m}(q^2) = q^{-n} F_{n,m-1}(q^2) + q^m F_{n-1,m}(q^2) \quad \forall n, m \geq 1.$$

5.2 Recurrence relation as a sum of sl_2-strings.

Now we try to write $F_{n,m}(q^2)$ as a sum of sl_2-strings. We will first state the recurrence relation, and then give a sketch of the proof of the relation.

Theorem 5.2. *$F_{n,m}(q^2)$ follows the following recursive form:*

(1) $F_{0,m}(q^2)$ and $F_{n,0}(q^2)$:

$$F_{0,m}(q^2) = F_{n,0}(q^2) = [1]_{q,1/q}, \quad \forall n, m \geq 0.$$

(2) Suppose that

$$F_{n,m-1}(q^2) = \sum_{i, x_i < n} [x_i]_{q,1/q} + \sum_{i, x_i \geq n} [x_j]_{q,1/q}$$

and

$$F_{n-1,m}(q^2) = \sum_{j, y_j \leq m} [y_j]_{q,1/q} + \sum_{j, y_j > m} [y_j]_{q,1/q},$$

then

$$F_{n,m}(q^2) = A + B - C,$$

where

$$A = \sum_{i,x_i<n} [x_i + n]_{q,1/q} + \sum_{i,x_i \geq n} [x_i + n]_{q,1/q},$$

$$B = \sum_{j,y_j>m} [y_j - m]_{q,1/q},$$

and

$$C = \sum_{i,x_i<n} [n - x_i]_{q,1/q}$$

It is easy to check that the case of $k = 2$((3.1) and (3.1)) and $k = 3$ (Theorem 4.1) are special cases of Theorem 5.2.

Following is a few examples according to the theorem:

Example 5.3.

$$F_{0,m}(q^2) = F_{n,0}(q^2) = [1]_{q,1/q}$$

$$F_{1,m}(q^2) = [m+2]_{q,1/q}$$

$$F_{2,1}(q^2) = [1+2]_{q,1/q} + [2-1]_{q,1/q} - [2-1]_{q,1/q} = [3]_{q,1/q}$$

$$F_{2,2}(q^2) = [3+2]_{q,1/q} + [3-2]_{q,1/q} = [5]_{q,1/q} + [1]_{q,1/q}$$

$$F_{2,3}(q^2) = [5+3]_{q,1/q} + [1+2]_{q,1/q} + [4-3]_{q,1/q} - [2-1]_{q,1/q} = [7]_{q,1/q} + [3]_{q,1/q}$$

$$F_{3,1}(q^2) = [1+3]_{q,1/q} + [3-1]_{q,1/q} - [3-1]_{q,1/q} = [4]_{q,1/q}$$

$$F_{3,2}(q^2) = [4+3]_{q,1/q} + [5-2]_{q,1/q} = [7]_{q,1/q} + [3]_{q,1/q}$$

$$F_{3,3}(q^2) = [7+3]_{q,1/q} + [3+3]_{q,1/q} + [7-3]_{q,1/q} = [10]_{q,1/q} + [6]_{q,1/q} + [4]_{q,1/q}$$

$$F_{4,1}(q^2) = [1+4]_{q,1/q} + [4-1]_{q,1/q} - [4-1]_{q,1/q} = [5]_{q,1/q}$$

$$F_{4,2}(q^2) = [5+4]_{q,1/q} + [7-2]_{q,1/q} + [3-2]_{q,1/q} = [9]_{q,1/q} + [5]_{q,1/q} + [1]_{q,1/q}$$

$$\begin{aligned} F_{4,3}(q^2) = {} & [9+4]_{q,1/q} + [5+4]_{q,1/q} + [1+4]_{q,1/q} + [10-3]_{q,1/q} + [6-3]_{q,1/q} + [4-3]_{q,1/q} \\ & - [4-1]_{q,1/q} = [13]_{q,1/q} + [9]_{q,1/q} + [7]_{q,1/q} + [5]_{q,1/q} + [1]_{q,1/q} \end{aligned}$$

Now we give the sketch of the proof to Theorem 5.2.

Proof. (Sketch)

Similar to the proof of Theorem 4.1, we Let $F'_{n,m}(q^2)$ be the rational function of q defined by the initial values and recurrence relation of Theorem 5.2 and Try to prove that $F'_{n,m}(q^2) = F_{n,m}(q^2)$ for all $n, m \geq 0$. Note that both $F'_{n,m}(q^2)$ and $F_{n,m}(q^2)$

can be written as the sum of sl_2-strings (and hence so is $F_{n,m}(q^2) - F'_{n,m}(q^2)$), it is sufficient to prove that $F_{n,m}(q^2) - F'_{n,m}(q^2)$ is a polynomial with constant term 0.

We prove this by induction. It is easy to show that $F_{0,m}(q^2) - F'_{0,m(q^2)} = 0$, $F_{n,0}(q^2) - F'_{n,0}(q^2) = 0$. Now we suppose that we have $F_{a,b}(q^2) = F'_{a,b}(q^2)$ for all $a+b < m+n$, and our goal is to prove that $F_{n,m}(q^2) - F'_{n,m}(q^2)$ is a polynomial with constant term 0.

Let $X = F_{n,m-1}(q^2)$, $Y = F_{n-1,m}(q^2)$, A, B, C are as defined in Theorem 4.1. Then $F_{n,m}(q^2) = q^{-n}X + q^m Y$, $F_{n-1,m}(q^2) = A + B - C$.

(1) We first prove that $(q^{-n}X + C) - A$ is a polynomial with constant term 0.

As a notation, let

$$X = \sum_{i,x_i<n} [x_i]_{q,1/q} + \sum_{i,x_i \geq n} [x_j]_{q,1/q} = X_1 + X_2,$$

$$A = \sum_{i,x_i<n} [x_i + n]_{q,1/q} + \sum_{i,x_i \geq n} [x_i + n]_{q,1/q} = A_1 + A_2.$$

(1a) Consider each sl_2-string $[x_i]$ in $X_2 = \sum_{i,x_i \geq n} [x_i]_{q,1/q}$. In this case we can prove that $q^{-n}[x_i]_{q,1/q} - [x_i + n]_{q,1/q}$ is a polynomial with constant term 0. Hence

$$q^{-n}X_2 - A_2 \tag{5.3}$$

is a polynomial with constant term 0.

(1b) For the sl_2-string $[x_i]$ in $X_1 = \sum_{i,x_i<n} [x_i]_{q,1/q}$, $q^{-n}[x_i]_{q,1/q}$ has only the terms

of $[x_i+n]_{q,1/q}$ for q^l of all $l \leq -(n-x_i)$, while $[n-x_i]_{q,1/q}$ has the terms of $[x_i+n]_{q,1/q}$ or q^l of all $-(n-x_i) < l \leq 0$. This give $q^{-n}[x_i]_{q,1/q} + [n-x_i]_{q,1/q} - [x_i+n]_{q,1/q}$ is a polynomial with constant term 0. Hence

$$q^{-n}X_1 + C - A_1 \tag{5.4}$$

is a polynomial with constant term 0.

(5.3) + (5.4) we have $(q^{-n}X + C) - A$ is a polynomial with constant term 0.

(2) Next we prove that $q^m Y - B$ is a polynomial with constant term 0.

Similarly, let

$$Y = \sum_{j, y_j \leq m} [y_j]_{q,1/q} + \sum_{j, y_j > m} [y_j]_{q,1/q} = Y_1 + Y_2.$$

(2a) For each sl_2-string $[y_j]_{q,1/q}$ in $Y_1 = \sum_{j,y_j \leq m}[y_j]_{q,1/q}$, $q^m[y_j]_{q,1/q}$ itself is a polynomial with constant term 0. Hence

$$q^m Y_1 \tag{5.5}$$

is a polynomial with constant term 0.

(2b) For each sl_2-string $[y_j]_{q,1/q}$ in $Y_2 = \sum_{j,y_j > m}[y_j]_{q,1/q}$, $q^m[y_j]_{q,1/q} - [y_j - m]_{q,1/q}$ is a polynomial with constant term 0. Hence

$$q^m Y_2 - B \tag{5.6}$$

is a polynomial with constant term 0.

(5.5) + (5.6) we have $q^m Y - B$ is a polynomial with constant term 0.

Now by (1) and (2), we have

$$q^{-n}X + C - A + q^m Y - B = F_{n,m}(q^2) - F'_{n,m}(q^2)$$

is a polynomial with constant term 0, and the proof is completed. $\square$

6 Breaking down the area statistic

For the second assumption

$$B_{n,k}(q,t) = \sum_{\pi \in \mathcal{E}_{n,k}} t^{\text{area}(\pi)},$$

we want to find a way to calculate the area statistic and rewrite as a sum of sl_2-strings. There are different ways to break down the problem. Base on the result of the recursive form of $B_3, k(1,t)$ and $B_{n,k}(q,1/q)$ in Section 5 and 6, one possible way is to break down the problem as following.

See an example of $\mathcal{E}_{4,6}$ first:

Example 6.1. We write the area statistic as the sum of the following:

(1) Set of paths X: path never touch the diagonal $y = x$ except the endpoint $(4,4)$. Then the path is in the following region (Figure 7, wide line), which gives that

$$\sum_{\pi \in X} t^{\text{area}} = t^4 \sum_{\pi \in \mathcal{E}_{4,5}} t^{\text{area}(\pi)} = t^4 B_{4,5}(1,t).$$

(2) Set of path $Y_m (m = 0,1,2,3)$: path passing (m,m) but not (a,a) for any $m <$

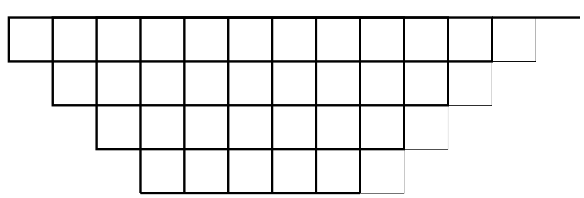


Figure 7: Path never touch $y = x$ except the endpoint.

$a < k$. For example, Y_1 is in the following region (Figure 8). As shown in the graph,

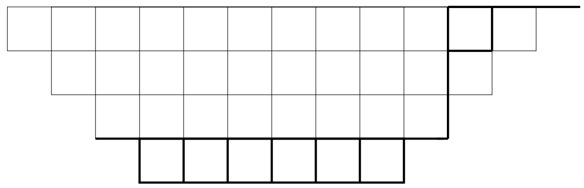


Figure 8: Path last touching diagonal at $(1, 1)$.

we can conclude that

$$\sum_{\pi \in Y_i} t^{\text{area}(\pi)} = t^{4-1-i} (\sum_{\pi \in \mathcal{E}_{i,6}} t^{\text{area}(\pi)}) (\sum_{\pi \in \mathcal{D}_{4-2-i}} t^{\text{area}(\pi)}) = t^{4-1-i} B_{i,6}(1, t) C_{4-2-i}(1, t)$$

Adding (1) and (2), we have

$$B_{4,6}(1, t) = t^4 B_{4,5}(1, t) + t^3 C_2(1, t) + t^2 B_{1,6}(1, t) C_1(1, t) + t B_{2,6}(1, t) + B_{3,6}(1, t).$$

Similarly, for any $n, \geq 2, k \geq 1$, we conclude that

$$B_{n,k}(1, t) = t^n B_{n,k-1}(1, t) + \sum_{i=0}^{n-1} t^{n-i-1} B_{i,k}(1, t) C_{n-2-i}(1, t),$$

where we define $C_{-1}(1, t) = C_0(1, t) = 1$ and $B_{0,k}(1, t) = 1$, $B_{n,0}(1, t) = B_{n-1,2}(1, t)$ here.

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