AMCS Written Preliminary Exam
Part II, May 2, 2014

1. Let A be a real $n \times n$ matrix of rank m. Show that $A^t A$ and AA^t also have rank m.
2. Let $0 < x_0 < \pi$, and define the sequence $x_{n+1} = \sin x_n$. Prove that $< x_n >$ converges and compute the limit $L = \lim_{n\to\infty} x_n$.
3. Let Γ be a simple closed $\mathscr{C}^1$-curve not passing through 0. Find all possible values for the integral
$$\int_\Gamma e^{\frac{1}{z}} dz.$$
4. Find a one-to-one conformal map from the half disk
$$D^+ = \{z :\ 0 < \operatorname{Im} z \text{ and } |z| < 1\}$$
onto $\mathbb{C} \setminus (-\infty, 0]$.
5. A stick of length 1 is broken into three pieces randomly. The two break points are chosen uniformly and independently. What is the probability that these pieces can form a triangle?
6. Prove that the series
$$\sum_{n=0}^{\infty} x^n(1 - x^n)$$
converges pointwise, but not uniformly on [0, 1]. Find a necessary and sufficient condition on α so that the series
$$\sum_{n=0}^{\infty} (1 - x)^\alpha x^n(1 - x^n)$$
does converge uniformly in [0, 1].
7. Let A_n be a sequence of square matrices converging to A. Give a proof or counterexample for the following statements:
   (a) If each A_n is singular, then A is singular as well.
   (b) If each A_n is non-singular, then A is non-singular as well.