AMCS Written Preliminary Exam
Part I, May 2, 2014

1. Show that the sum
$$f(x) = \sum_{n=1}^{\infty} \log\left[1 + \left(\frac{x}{n}\right)^2\right]$$
converges uniformly for x in any finite interval $(-N, N)$. Show that $f(x)$ is differentiable, and give a formula for its derivative (possibly as a sum)?

2. Suppose that $f(z)$ is an entire function (analytic in all of $\mathbb{C}$), and let
$$M(R) = \max\{|f(z)| : |z| = R\}. \tag{1}$$
Suppose that $M(2R) \leq 2^N M(R)$. Show that f is a polynomial of degree at most N.

3. Suppose that $f(z) = z^n + z^3 + z + 2$. Show that all roots of f tend to the unit circle as $n \to \infty$.

4. For which real values of a does
$$\lim_{\epsilon \to 0^+} \int_{\epsilon}^{\frac{1}{2}} \frac{dx}{x|\log x|^a} \tag{2}$$
exist?

5. Let
$$A = \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix} \quad B = \begin{pmatrix} 4 & 1 \\ 1 & 4 \end{pmatrix} \text{ and } x = \begin{pmatrix} 3 \\ 2 \end{pmatrix}. \tag{3}$$
What is
$$\lim_{N \to \infty} \frac{\|A^N x\|}{\|B^N x\|}? \tag{4}$$

6. Find all 3×3 matrices A with eigenvalues 2 and 3 (and no others) such that the equations
$$(A - 2\,\mathrm{Id})v = 0 \text{ and } (A - 3\,\mathrm{Id})w = 0 \tag{5}$$
each have a 1-dimensional solution space.

7. Let X_1 be a random variable uniformly distributed in $[0, 1]$. For $n \geq 1$, let X_{n+1} be equal to X_n times an independent random variable chosen uniformly on $[0, 1]$.
   (a) Compute EX_n.
   (b) What is the limit as $n \to \infty$ of $P(X_n > (0.25)^n)$? Hint: Take a log.