AMCS Written Preliminary Exam
Part II, August 27, 2014

1. Define a sequence of functions on $\mathbb{R}$ by setting:
$$f_n(x) = \frac{x}{1+nx^2}$$
Prove that $< f_n(x) >$ converges uniformly on $\mathbb{R}$ to a function $f(x)$. For which x is it true that
$$\lim_{n\to\infty} f_n'(x) = f'(x)?$$

2. Suppose that $f(z)$ is analytic in the disk of radius 2. What is the value of the contour integral
$$\int\limits_{|z|=1} f\left(\frac{1}{z}\right) dz?$$

3. Consider the following game of chance: A circular target of radius 1 is divided into n concentric circles of radius $1/n, 2/n, ..., n/n = 1$. A dart is tossed at random onto the circle; if it lands in the annular zone between the circles with radii k/n and $(k+1)/n$, then $n-k$ dollars are won, with $k = 0, \ldots, n-1$. Let X_n be a random variable denoting the amount of money won in one round of the game, and $E(X_n)$ its expected value. Compute the limit
$$\lim_{n\to\infty} \frac{E(X_n)}{n}.$$

4. Suppose that A is an $n \times n$ skew-symmetric matrix ($A^t = -A$). Prove that if n is an odd number, then there is a non-zero vector v_0 such that $Av_0 = 0$. Show that e^A is an orthogonal matrix, that is
$$\langle e^A x, e^A y\rangle = \langle x, y\rangle,$$
for any pair of vectors $x, y \in \mathbb{R}^n$. What is $e^A v_0$?

5. Suppose A is a symmetric, positive definite $n \times n$ matrix. Compute the integral
$$\int_{\mathbb{R}^n} e^{-\frac{1}{2}\langle Ax,x\rangle} dx.$$
Note that
$$\int\limits_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}. \tag{1}$$
Hint: First diagonalize A.

6. A Markov process with 2 states $\{A,\ B\}$ is defined by the following transition probabilities:

$$\text{Prob}(A|A) = \frac{1}{3} \qquad \text{Prob}(B|A) = \frac{2}{3}$$
$$\text{Prob}(A|B) = \frac{1}{2} \qquad \text{Prob}(B|B) = \frac{1}{2}.$$

After many, many transitions what is the probability that the system will be found in state B?