1. Let

\[
 f(x) = \begin{cases} 
 (e^{x^2} - e^{-x^2}) \sin \left( \frac{1}{x^3} \right) & \text{for } x \neq 0 \\
 0 & \text{for } x = 0 
\end{cases}
\]

Show that \( f \) is differentiable at 0 and compute \( f'(0) \).

2. Give an example of a Riemann integrable function with a countably infinite number of points of discontinuity.

3. For which real values of \( p \) is the following series convergent?

\[
 \sum_{n=2}^{\infty} \frac{1}{(n^2 \log n) \cdot (n^{\frac{1}{3}} - 1)^p}
\]

4. Suppose that \( A \) is an \( n \times n \) matrix with a 1-dimensional null-space. Show that we can choose vectors \( u \) and \( v \) so that the linear transformation

\[
 B = A + u \otimes v^t
\]

is invertible. Here \( u \otimes v^t x \xrightarrow{d} (x, v)u \). What conditions must \( u \) and \( v \) satisfy for this to be true?

5. Let \( A \) be the \( n \times n \) matrix:

\[
 A = \begin{bmatrix}
 0 & 2 & \ldots & 2 & 2 \\
 2 & 0 & \ldots & 2 & 2 \\
 \vdots & \ddots & \ddots & 0 & 2 \\
 2 & \ldots & 2 & 0 
\end{bmatrix}
\]

(0s along the diagonal and 2s everywhere else.) Compute the eigenvalues and eigenvectors of \( A \).

6. A real valued \( \mathbb{C}^2 \)-function \( u \), defined in the unit disk, is harmonic if it satisfies the partial differential equation \( \partial_{xx}u + \partial_{yy}u = 0 \). Show that every such function can expressed in the form

\[
 u(x, y) = f(x + iy) + \overline{f(x + iy)},
\]

where \( f \) is an analytic function in the unit disk.
7. A fair coin is flipped until it comes up HEADS. If this occurs on the $n^{th}$ flip, then you win $\left(\frac{1}{2}\right)^n$. What is your expected winnings?

8. Let $\{Y_n\}$ be independent random variables uniformly distributed in $[0, 1]$. For $n \geq 1$, let $X_n = Y_1 \cdots Y_n$, the products of the $Y_i$.
   
   (a) Compute $E[X_n]$.
   
   (b) What is the limit, as $n \to \infty$, of $\text{Prob}(X_n > (0.4)^n)$?